

Quantum Isometry groups of dual of finitely generated discrete groups and quantum groups

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Abstract

We study quantum isometry groups, denoted by $\mathbb{Q}(\Gamma, S)$, of spectral triples on $C_r^*(\Gamma)$ for a finitely generated discrete group Γ coming from the word-length metric with respect to a symmetric generating set S . We first prove a few general results about $\mathbb{Q}(\Gamma, S)$ including:

- For a group Γ with polynomial growth property, the dual of $\mathbb{Q}(\Gamma, S)$ has polynomial growth property provided the action of $\mathbb{Q}(\Gamma, S)$ on $C_r^*(\Gamma)$ has full spectrum.
- $\mathbb{Q}(\Gamma, S) \cong QISO(\hat{\Gamma}, d)$ for any discrete abelian group Γ , where d is a suitable metric on the dual compact abelian group $\hat{\Gamma}$.

We then carry out explicit computations of $\mathbb{Q}(\Gamma, S)$ for several classes of examples including free and direct product of cyclic groups, Baumslag-Solitar group, Coxeter groups etc. In particular, we have computed quantum isometry groups of all finitely generated abelian groups which do not have factors of the form \mathbb{Z}_2^k or \mathbb{Z}_4^l for some k, l in the direct product decomposition into cyclic subgroups.

1 Introduction

It is a very interesting problem, both from the physical and mathematical viewpoint, to understand and classify quantum symmetries of possibly non-commutative C^* -algebras (usually with further structures), i.e. possible actions of quantum groups on them. In [19], this problem was considered in an algebraic and categorical setting, leading to the realization of some of the well known (algebraic) quantum groups such as $SL(2, q)$ as the universal object in some category of quantum groups acting on the quantum 2-plane. S.Wang [23] took up a similar problem in the analytical framework of compact quantum groups acting on C^* -algebras. Later on, a number of mathematicians including Wang, Banica, Bichon and others ([23], [1], [8]) developed a theory of quantum automorphism groups of finite dimensional C^* -algebras as well as quantum isometry groups of finite metric spaces and finite graphs. In [14] the first named author of the present article extended such constructions to the set up of possibly infinite dimensional C^* -algebras, and more interestingly, that of spectral triples a la Connes [11], by defining and studying quantum isometry groups of spectral triples. This led to the study of such quantum isometry groups by many authors including Goswami, Bhowmick, Skalski, Banica, Bichon, Soltan,

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Das, Joardar and others. In the present article, our focus is on a rather special yet interesting and important class of spectral triples, namely those coming from the word-length metric of finitely generated discrete groups with respect to some given symmetric generating set. There have been several articles already on computations and study of the quantum isometry groups of such spectral triples, e.g [7], [22], [13], [4], [2] and references therein. However, for a systematic and unified study of quantum isometry groups of such spectral triples, one needs to look at many more examples and then try to identify some general pattern. This is the main objective of this paper. We have not yet been able to propose a general theory, but almost complete the understanding of quantum isometry groups of direct and free product of cyclic groups, except a few cases only. Besides, we treat several other important classes of groups.

We begin by proving some general facts about the quantum isometry group $\mathbb{Q}(\Gamma, S)$ of a discrete group Γ with a finite symmetric generating set S . We prove, among other things, the following two interesting results:

1. If Γ has polynomial growth and the action of $\mathbb{Q}(\Gamma, S)$ on $C_r^*(\Gamma)$ has full spectrum, then the dual (discrete quantum group) of $\mathbb{Q}(\Gamma)$ has polynomial growth.
2. In case Γ is abelian, there is a metric on the dual compact abelian group $\hat{\Gamma}$ such that the corresponding quantum isometry group in the metric space sense (as in [15]) exists and coincides with $\mathbb{Q}(\Gamma, S)$.

Next we carry out several explicit computations. We have given special emphasis on groups of the form $\Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ or $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$, where $\Gamma_i = \mathbb{Z}_{n_i}$ for some n_i . We have proved that in many cases the quantum isometry groups of these groups turn out to be the free or tensor product of the quantum isometry groups of the factors Γ_i 's. Here is a brief list of groups for which we have computed the quantum isometry groups in this paper:

- All finitely generated abelian groups which do not have factors of the form \mathbb{Z}_2 or \mathbb{Z}_4 in the direct product decomposition into cyclic subgroups.
- Free product of all cyclic groups which do not have factors of the form \mathbb{Z}_2 or \mathbb{Z}_4 .
- Some special cases of direct or free product of cyclic groups having \mathbb{Z}_2 or \mathbb{Z}_4 as factors.
- Coxeter group.
- Baumslag-Solitar group.
- Dihedral, Tetrahedral, Icosahedral and Octahedral groups.

2 Quantum isometry group of $C_r^*(\Gamma)$: existence and some generalities

We begin with a few basic definitions and facts about quantum isometry groups of spectral triples defined by Bhowmick and Goswami in [6]. We denote the algebraic tensor product, spatial (minimal) C^* -tensor product and maximal C^* -tensor product by \otimes , $\hat{\otimes}$ and \otimes^{max} respectively. We'll use the leg-numbering notation. Let \mathcal{H} be a complex Hilbert space, $\mathcal{K}(\mathcal{H})$ the C^* algebra of compact operators on it, and \mathcal{Q} a unital C^* algebra. The multiplier algebra $\mathcal{M}(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ has two natural embeddings into $\mathcal{M}(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q} \hat{\otimes} \mathcal{Q})$, one obtained by extending the map $x \mapsto x \otimes 1$ and the second one is obtained by composing this map with the flip on the last two factors. We will write ω^{12} and ω^{13} for the images of an element $\omega \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ under these two maps respectively. We'll denote by $\mathcal{H} \bar{\otimes} \mathcal{Q}$ the Hilbert C^* -module obtained by completing $\mathcal{H} \otimes \mathcal{Q}$ with respect to the norm induced by the \mathcal{Q} valued inner product $\langle\langle \xi \otimes q, \xi' \otimes q' \rangle\rangle := \langle \xi, \xi' \rangle q^* q'$, where $\xi, \xi' \in \mathcal{H}$ and $q, q' \in \mathcal{Q}$.

2.1 Basic definitions

Definition 2.1 A compact quantum group (CQG for short) is a pair (\mathcal{Q}, Δ) , where \mathcal{Q} is a unital C^* -algebra and $\Delta : \mathcal{Q} \rightarrow \mathcal{Q} \hat{\otimes} \mathcal{Q}$ is a unital C^* -homomorphism satisfying the following two conditions:

1. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ (co-associativity).
2. Each of the linear spans of $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})$ and that of $\Delta(\mathcal{Q})(\mathcal{Q} \otimes 1)$ is norm-dense in $\mathcal{Q} \hat{\otimes} \mathcal{Q}$.

A CQG morphism from $(\mathcal{Q}_1, \Delta_1)$ to another $(\mathcal{Q}_2, \Delta_2)$ is a unital C^* -homomorphism $\pi : \mathcal{Q}_1 \mapsto \mathcal{Q}_2$ such that $(\pi \otimes \pi)\Delta_1 = \Delta_2\pi$.

Definition 2.2 $(\mathcal{Q}_1, \Delta_1)$ is called a quantum subgroup of $(\mathcal{Q}_2, \Delta_2)$ if there exists a surjective C^* -homomorphism η from \mathcal{Q}_2 to \mathcal{Q}_1 such that $(\eta \otimes \eta)\Delta_2 = \Delta_1\eta$ holds.

Sometimes we may denote the CQG (\mathcal{Q}, Δ) simply as \mathcal{Q} , if Δ is clear from the context.

Definition 2.3 A unitary (co)representation of (\mathcal{Q}, Δ) on a Hilbert space \mathcal{H} is a \mathbb{C} -linear map from \mathcal{H} to the Hilbert module $\mathcal{H} \bar{\otimes} \mathcal{Q}$ such that

1. $\langle\langle U(\xi), U(\eta) \rangle\rangle = \langle \xi, \eta \rangle 1_{\mathcal{Q}}$ (for all $\xi, \eta \in \mathcal{H}$).
2. $(U \otimes id)U = (id \otimes \Delta)U$.
3. Span $\{U(\xi)b : \xi \in \mathcal{H}, b \in \mathcal{Q}\}$ is dense in $\mathcal{H} \bar{\otimes} \mathcal{Q}$.

Given such a unitary representation we have a unitary element \tilde{U} belonging to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ given by $\tilde{U}(\xi \otimes b) = U(\xi)b$, ($\xi \in \mathcal{H}$, $b \in \mathcal{Q}$) satisfying $(id \otimes \Delta)(\tilde{U}) = \tilde{U}^{12} \tilde{U}^{13}$.

Remark 2.4 *It is known that the linear span of matrix elements of a finite dimensional unitary representation forms a dense Hopf *-algebra \mathcal{Q}_0 of (\mathcal{Q}, Δ) , on which an antipode κ and co-unit ϵ are defined.*

Definition 2.5 *A closed subspace \mathcal{H}_1 of \mathcal{H} is said to be invariant if $U(\mathcal{H}_1) \subseteq \mathcal{H}_1 \hat{\otimes} \mathcal{Q}$. A unitary representation U of a CQG is said to be irreducible if there is no proper invariant subspace.*

We denote by $\hat{\mathcal{Q}}$ the set of inequivalent irreducible representations of (\mathcal{Q}, Δ) . For $\pi \in \hat{\mathcal{Q}}$, let d_π and $\{t_{jk}^\pi : j, k = 1, \dots, d_\pi\}$ be the dimension and matrix coefficients of the corresponding finite dimensional representation respectively w.r.t some basis $\{e_1, e_2, \dots, e_{d_\pi}\}$, i.e. $\pi(e_i) = \sum_{j=1}^{d_\pi} e_j \otimes t_{ij}^\pi$. Note that $\mathcal{Q}_0 = \text{span}\{t_{ij}^\pi \mid \forall \pi \in \hat{\mathcal{Q}}\}$. The coproduct of \mathcal{Q} is given by $\Delta(t_{ij}^\pi) = \sum_{k=1}^{d_\pi} t_{kj}^\pi \otimes t_{ik}^\pi$. Then for each $\pi \in \hat{\mathcal{Q}}$, we have a unique $d_\pi \times d_\pi$ complex matrix F_π such that

1. F_π is positive and invertible with $Tr(F_\pi) = Tr(F_\pi^{-1}) = M_\pi > 0$ (say).
2. $h(t_{ij}^\pi t_{kl}^{\pi*}) = 1/M_\pi \delta_{ik} F_\pi(j, l)$.

Corresponding to $\pi \in \hat{\mathcal{Q}}$, let ρ_{sm}^π be the linear functional on \mathcal{Q} given by $\rho_{sm}^\pi(x) = h(x_{sm}^\pi x)$, $s, m = 1, \dots, d_\pi$ for $x \in \mathcal{Q}$, where $x_{sm}^\pi = (M_\pi) t_{km}^{\pi*} (F_\pi)_{ks}$. Also let $\rho^\pi = \sum_{s=1}^{d_\pi} \rho_{ss}^\pi$.

Definition 2.6 *We say that a CQG (\mathcal{Q}, Δ) acts on a unital C^* -algebra B if there is a unital C^* -homomorphism (called action) $\alpha : B \rightarrow B \hat{\otimes} \mathcal{Q}$ satisfying the following:*

1. $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$.
2. Linear span of $\alpha(B)(1 \otimes \mathcal{Q})$ is norm-dense in $B \hat{\otimes} \mathcal{Q}$.

Definition 2.7 *The action is said to be faithful if the *-algebra generated by the set $\{(f \otimes id)\alpha(b) \mid f \in B^*, \forall b \in B\}$ is norm-dense in \mathcal{Q} , where B^* is the Banach space dual of B .*

Remark 2.8 *Given an action α of a CQG \mathcal{Q} on a unital C^* -algebra B , we can always find a norm-dense, unital *-subalgebra $B_0 \subseteq B$ such that $\alpha|_{B_0} : B_0 \mapsto B_0 \otimes \mathcal{Q}_0$ is a Hopf-algebraic co-action. Moreover, α is faithful if and only if the *-algebra generated by $\{(f \otimes id)\alpha(b) \mid f \in B_0^*, \forall b \in B_0\}$ is the whole of \mathcal{Q}_0 .*

Now we can define a projection $P_\pi : B \rightarrow B$ by $P_\pi := (id \otimes \rho^\pi)\alpha$ (note that $(id \otimes \phi)\alpha(B) \subseteq B$ for all bounded linear functionals ϕ on \mathcal{Q}). We denote $Im(P_\pi)$ by B_π , and call it the spectral subspace coming from π . We say that the action

is of full spectrum if $B_\pi \neq 0 \forall \pi \in \hat{\mathcal{Q}}$.

Given two CQG's $\mathcal{Q}_1, \mathcal{Q}_2$ the free product $\mathcal{Q}_1 \star \mathcal{Q}_2$ as well as the maximal tensor product $\mathcal{Q}_1 \otimes^{\max} \mathcal{Q}_2$ admit the natural CQG structures, as given in [24], [25]. Moreover, they have the following universal properties (see [24], [25]).

- Proposition 2.9** 1. *The canonical injections, say i_1, i_2 (j_1, j_2 respectively) from \mathcal{Q}_1 and \mathcal{Q}_2 to $\mathcal{Q}_1 \star \mathcal{Q}_2$ ($\mathcal{Q}_1 \otimes^{\max} \mathcal{Q}_2$ respectively) are CQG morphisms.*
2. *Given any CQG \mathcal{C} and morphisms $\pi_1 : \mathcal{Q}_1 \mapsto \mathcal{C}$ and $\pi_2 : \mathcal{Q}_2 \mapsto \mathcal{C}$ there always exists a unique morphism denoted by $\pi := \pi_1 \star \pi_2$ from $\mathcal{Q}_1 \star \mathcal{Q}_2$ to \mathcal{C} satisfying $\pi \circ i_k = \pi_k$ for $k = 1, 2$.*
3. *Furthermore, if the ranges of π_1 and π_2 commute, i.e. $\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a) \forall a \in \mathcal{Q}_1, b \in \mathcal{Q}_2$, we have a unique morphism π' from $\mathcal{Q}_1 \otimes^{\max} \mathcal{Q}_2$ to \mathcal{C} satisfying $\pi' \circ j_k = \pi_k$ for $k = 1, 2$.*
4. *The above conclusions hold for free or maximal tensor product of any finite number of CQG's as well.*

Definition 2.10 Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type (a la Connes). Consider the category $Q(\mathcal{D}) \equiv Q(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ whose objects are (\mathcal{Q}, U) , where (\mathcal{Q}, Δ) is a CQG having a unitary representation U on the Hilbert space \mathcal{H} satisfying the following:

1. \tilde{U} commutes with $(\mathcal{D} \otimes 1_{\mathcal{Q}})$.
2. $(id \otimes \phi) \circ ad_{\tilde{U}}(a) \in (\mathcal{A}^\infty)''$ for all $a \in \mathcal{A}^\infty$ and ϕ is any state on \mathcal{Q} , where $ad_{\tilde{U}}(x) := \tilde{U}(x \otimes 1)\tilde{U}^*$ for $x \in \mathcal{B}(\mathcal{H})$.

A morphism between two such objects (\mathcal{Q}, U) and (\mathcal{Q}', U') is a CQG morphism $\psi : \mathcal{Q} \rightarrow \mathcal{Q}'$ such that $U' = (id \otimes \psi)U$. If a universal object exists in $Q(\mathcal{D})$ then we denote it by $\widetilde{QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})}$ and the corresponding largest Woronowicz subalgebra for which $ad_{\tilde{U}_0}$ is faithful, where U_0 is the unitary representation of $\widetilde{QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})}$, is called the quantum group of orientation preserving isometries and denoted by $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

Let us state Theorem 2.23 of [6] which gives a sufficient condition for the existence of $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

Theorem 2.11 Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type. Assume that \mathcal{D} has one dimensional kernel spanned by a vector $\xi \in \mathcal{H}$ which is cyclic and separating for \mathcal{A}^∞ and each eigenvector of \mathcal{D} belongs to $\mathcal{A}^\infty \xi$. Then $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ exists.

Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple satisfying the condition of Theorem 2.11 and $\mathcal{A}_{00} = Lin\{a \in \mathcal{A}^\infty : a\xi \text{ is an eigenvector of } \mathcal{D}\}$. Moreover, assume that \mathcal{A}_{00} is norm-dense in \mathcal{A}^∞ . Let $\hat{\mathcal{D}} : \mathcal{A}_{00} \mapsto \mathcal{A}_{00}$ be defined by $\hat{\mathcal{D}}(a)\xi = \mathcal{D}(a\xi)$ ($a \in \mathcal{A}_{00}$). This is well defined as ξ is cyclic and separating vector for \mathcal{A}^∞ . Let τ be the vector state corresponding to the vector ξ .

Definition 2.12 Let \mathcal{A} be a C^* -algebra and \mathcal{A}^∞ be a dense $*$ -subalgebra such that $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ is a spectral triple as above. Let $\hat{\mathbf{C}}(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be the category with objects (\mathcal{Q}, α) such that \mathcal{Q} is a CQG with a C^* -action α on \mathcal{A} such that

1. α is τ preserving, i.e. $(\tau \otimes \text{id})\alpha(a) = \tau(a).1$ for all $a \in \mathcal{A}$.
2. α maps \mathcal{A}_{00} into $\mathcal{A}_{00} \otimes \mathcal{Q}$.
3. $\alpha\hat{\mathcal{D}} = (\hat{\mathcal{D}} \otimes I)\alpha$.

The morphisms in $\hat{\mathbf{C}}(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ are CQG morphisms intertwining the respective actions.

Proposition 2.13 It is shown in Corollary 2.27 of [6] that $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ is the universal object in $\hat{\mathbf{C}}(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

2.2 QISO for a spectral triple on $C_r^*(\Gamma)$

Now we discuss the special case of our interest. Connes considered this spectral triple in [10]. Let Γ be a finitely generated discrete group with generating set $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_k, a_k^{-1}\}$. We make the convention of choosing the generating set to be symmetric, i.e. $a_i \in S$ implies $a_i^{-1} \in S \forall i$. In case some a_i has order 2, we include only a_i , i.e. not count it twice. The corresponding word length function on the group defined by $l(g) = \min \{r \in \mathbb{N}, g = h_1 h_2 \dots h_r\}$ where $h_i \in S$, i.e. for each i , $h_i = a_j$ or a_j^{-1} for some j . Notice that $S = \{g \in \Gamma \mid l(g) = 1\}$, using this length function we can define a metric on Γ by $d(a, b) = l(a^{-1}b) \forall a, b \in \Gamma$. This is called the word metric corresponding to the generating set S . Now consider the algebra $C_r^*(\Gamma)$, which is the C^* -completion of the group ring $\mathbb{C}\Gamma$ viewed as a subalgebra of $B(l^2(\Gamma))$ in the natural way via the left regular representation. We define a Dirac operator $D_\Gamma(\delta_g) = l(g)\delta_g$. In general, D_Γ is an unbounded operator.

$$\text{Dom}(D_\Gamma) = \{\xi \in l^2(\Gamma) : \sum_{g \in \Gamma} l(g)^2 |\xi(g)|^2 < \infty\}.$$

Here, δ_g is the vector in $l^2(\Gamma)$ which takes value 1 at the point g and 0 at all other points. Natural generators of the algebra $\mathbb{C}\Gamma$ (images in the left regular representation) will be denoted by λ_g , i.e. $\lambda_g(\delta_h) = \delta_{gh}$. Let us define

$$\Gamma_r = \{\delta_g \mid l(g) = r\},$$

$$\Gamma_{\leq r} = \{\delta_g \mid l(g) \leq r\}.$$

Moreover, let p_r and q_r be the orthogonal projections onto $Sp(\Gamma_r)$ and $Sp(\Gamma_{\leq r})$ respectively. Clearly

$$D_\Gamma = \sum_{n \in \mathbb{N}_0} n p_n,$$

where $p_r = q_r - q_{r-1}$ and $p_0 = q_0$. The canonical trace on $C_r^*(\Gamma)$ is given by $\tau(\sum_{g \in \Gamma} c_g \lambda_g) = c_e$. It is easy to check that $(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ is a spectral triple

using Lemma 1.1 of [20]. Now take $\mathcal{A} = C_r^*(\Gamma)$, $\mathcal{A}^\infty = \mathbb{C}\Gamma$, $\mathcal{H} = l^2(\Gamma)$ and $\mathcal{D} = D_\Gamma$ as before. Then $\text{QISO}^+(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ exists by Theorem 2.11, taking δ_e as the cyclic separating vector for $\mathbb{C}\Gamma$. As $\text{QISO}^+(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ depends on the generating set of Γ it is denoted by $\mathbb{Q}(\Gamma, S)$. Most of the times we denote it by $\mathbb{Q}(\Gamma)$ if S is understood from the context. Now as in [7] its action α (say) on $C_r^*(\Gamma)$ is determined by

$$\alpha(\lambda_\gamma) = \sum_{\gamma' \in S} \lambda_{\gamma'} \otimes q_{\gamma, \gamma'},$$

where the matrix $[q_{\gamma, \gamma'}]_{\gamma, \gamma' \in S}$ is called the fundamental representation in $M_{\text{card}(S)}(\mathbb{Q}(\Gamma, S))$. Note that we have $\Delta(q_{\gamma, \gamma'}) = \sum_\beta q_{\beta, \gamma'} \otimes q_{\gamma, \beta}$.

$\mathbb{Q}(\Gamma, S)$ is also the universal object in the category $\hat{\mathbf{C}}(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ by Proposition 2.13 and observe that all the eigenspaces of $\hat{\mathcal{D}}_\Gamma$, where \mathcal{D}_Γ is as in Definition 2.12, are invariant under the action. The eigenspaces of $\hat{\mathcal{D}}_\Gamma$ are precisely $\text{Span}\{\lambda_g \mid l(g) = r\}$ with $r \geq 0$.

It can also be identified with the universal object of some other categories naturally arising in the context. Consider the category \mathbf{C}_τ of CQG's consisting of the objects (\mathcal{Q}, α) such that α is an action of \mathcal{Q} on $C_r^*(\Gamma)$ satisfying the following two properties:

1. α leaves $Sp(\Gamma_1)$ invariant.
2. It preserves the canonical trace τ of $C_r^*(\Gamma)$.

Morphisms in \mathbf{C}_τ are CQG morphisms intertwining the respective actions.

Lemma 2.14 *The two categories \mathbf{C}_τ and $\hat{\mathbf{C}}(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ are isomorphic.*

For the proof the reader is referred to Lemma 2.16 of [18].

Corollary 2.15 *It follows from Lemma 2.14 that there is a universal object, say $(\mathcal{Q}_\tau, \alpha_\tau)$ in \mathbf{C}_τ and $(\mathcal{Q}_\tau, \alpha_\tau) \cong \mathbb{Q}(\Gamma, S)$.*

We now identify $\mathbb{Q}(\Gamma, S)$ as a universal object in yet another category. Let us recall the quantum free unitary group $A_u(n)$ introduced in [24]. It is the universal unital C^* -algebra generated by $((a_{ij}))$ subject to the conditions that $((a_{ij}))$ and $((a_{ji}))$ are unitaries. Moreover, it admits a co-product structure with comultiplication $\Delta(a_{ij}) = \sum_{l=1}^n a_{lj} \otimes a_{il}$. Consider the category \mathbf{C} with objects $(\mathcal{C}, \{x_{ij}, i, j = 1, \dots, 2k\})$ where \mathcal{C} is a unital C^* -algebra generated by $((x_{ij}))$ such that $((x_{ij}))$ as well as $((x_{ji}))$ are unitaries and there is a unital C^* -homomorphism $\alpha_{\mathcal{C}}$ from $C_r^*(\Gamma)$ to $C_r^*(\Gamma) \hat{\otimes} \mathcal{C}$ sending e_i to $\sum_{j=1}^{2k} e_j \otimes x_{ij}$, where $e_{2i-1} = \lambda_{a_i}$ and $e_{2i} = \lambda_{a_i}^{-1} \forall i = 1, \dots, k$. The morphisms from $(\mathcal{C}, \{x_{ij}, i, j = 1, \dots, 2k\})$ to $(\mathcal{P}, \{p_{ij}, i, j = 1, \dots, 2k\})$ are unital $*$ -homomorphisms $\beta : \mathcal{C} \mapsto \mathcal{P}$ such that $\beta(x_{ij}) = p_{ij}$.

Moreover, by definition of each object $(\mathcal{C}, \{x_{ij}, i, j = 1, \dots, 2k\})$ we get a unital $*$ -morphism $\rho_{\mathcal{C}}$ from $A_u(2k)$ to \mathcal{C} sending a_{ij} to x_{ij} . Let the kernel of this map

be \mathcal{I}_C and \mathcal{I} be intersection of all such ideals. Then $\mathcal{C}^{\mathcal{U}} := A_u(2k)/\mathcal{I}$ is the universal object generated by $x_{ij}^{\mathcal{U}}$ in the category \mathbf{C} . Furthermore, we can show, following a line of arguments similar to those in Theorem 4.8 of [15], that it has a CQG structure with the co-product $\Delta(x_{ij}^{\mathcal{U}}) = \sum_l x_{lj}^{\mathcal{U}} \otimes x_{il}^{\mathcal{U}}$.

Proposition 2.16 $(\mathcal{Q}_\tau, \alpha_\tau)$ and $\mathcal{C}^{\mathcal{U}}$ are isomorphic as CQG.

Proof:

Let $((q_{ij}))$, $1 \leq i, j \leq 2k$ be the fundamental representation of $(\mathcal{Q}_\tau, \alpha_\tau)$. Then we know that $((q_{ij}))$ and $((q_{ji}))$ are unitaries. By the universal property of $\mathcal{C}^{\mathcal{U}}$, we always get a surjective map from $\mathcal{C}^{\mathcal{U}}$ to $(\mathcal{Q}_\tau, \alpha_\tau)$ sending $x_{ij}^{\mathcal{U}}$ to q_{ij} , which intertwines the actions too.

On the other hand, we can construct a state on $C_r^*(\Gamma)$ defined by $\tilde{\tau} := (\tau \otimes h)\alpha_C^{\mathcal{U}}$, where h is the Haar state of $(\mathcal{C}^{\mathcal{U}}, x_{ij}^{\mathcal{U}})$, is clearly $\alpha_C^{\mathcal{U}}$ invariant. Note that $(C_r^*(\Gamma), \Delta_\Gamma, t_{ij})$ is an object in the category \mathbf{C} , where $((t_{ij}))$ is the diagonal matrix with entries $t_{2i-1, 2i-1} = \lambda_{a_i}$, $t_{2i, 2i} = \lambda_{a_i-1}$. By the universal property of $\mathcal{C}^{\mathcal{U}}$ we always get a surjective *- morphism π from $\mathcal{C}^{\mathcal{U}}$ to $C_r^*(\Gamma)$ sending $x_{ij}^{\mathcal{U}}$ to t_{ij} such that $(id \otimes \pi)\alpha_C^{\mathcal{U}} = \Delta_\Gamma$ holds. Moreover, we get

$$(\tilde{\tau} \otimes id)\Delta_\Gamma(x) = (id \otimes \tilde{\tau})\Delta_\Gamma(x) = \tilde{\tau}(x).1 \quad \forall x \in C_r^*(\Gamma).$$

But we know that the canonical trace (Haar state) τ is the unique bi-invariant state on $(C_r^*(\Gamma), \Delta_\Gamma)$. Hence, $\tilde{\tau} = \tau$. Then by universality of $(\mathcal{Q}_\tau, \alpha_\tau)$ we get a surjective map from $(\mathcal{Q}_\tau, \alpha_\tau)$ to $\mathcal{C}^{\mathcal{U}}$ sending q_{ij} to $x_{ij}^{\mathcal{U}}$, which also intertwines the actions. Thus, $(\mathcal{Q}_\tau, \alpha_\tau)$ and $\mathcal{C}^{\mathcal{U}}$ are isomorphic as CQG. \square

If Γ is commutative, the maximal commutative quantum subgroup of $\mathbb{Q}(\Gamma)$ (its abelianization) is denoted by $C(ISO(\Gamma))$. In Section 3 we will see many examples for which $\mathbb{Q}(\Gamma) \cong C(ISO(\Gamma))$. Now we fix some notational conventions

which will be useful in later sections. Note that the action α is of the form

$$\begin{aligned}
\alpha(\lambda_{a_1}) &= \lambda_{a_1} \otimes A_{11} + \lambda_{a_1^{-1}} \otimes A_{12} + \lambda_{a_2} \otimes A_{13} + \lambda_{a_2^{-1}} \otimes A_{14} + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{1(2k-1)} + \lambda_{a_k^{-1}} \otimes A_{1(2k)}, \\
\alpha(\lambda_{a_1^{-1}}) &= \lambda_{a_1} \otimes A_{12}^* + \lambda_{a_1^{-1}} \otimes A_{11}^* + \lambda_{a_2} \otimes A_{14}^* + \lambda_{a_2^{-1}} \otimes A_{13}^* + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{1(2k)}^* + \lambda_{a_k^{-1}} \otimes A_{1(2k-1)}^*, \\
\alpha(\lambda_{a_2}) &= \lambda_{a_1} \otimes A_{21} + \lambda_{a_1^{-1}} \otimes A_{22} + \lambda_{a_2} \otimes A_{23} + \lambda_{a_2^{-1}} \otimes A_{24} + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{2(2k-1)} + \lambda_{a_k^{-1}} \otimes A_{2(2k)}, \\
\alpha(\lambda_{a_2^{-1}}) &= \lambda_{a_1} \otimes A_{22}^* + \lambda_{a_1^{-1}} \otimes A_{21}^* + \lambda_{a_2} \otimes A_{24}^* + \lambda_{a_2^{-1}} \otimes A_{23}^* + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{2(2k)}^* + \lambda_{a_k^{-1}} \otimes A_{2(2k-1)}^*, \\
&\vdots \\
\alpha(\lambda_{a_k}) &= \lambda_{a_1} \otimes A_{k1} + \lambda_{a_1^{-1}} \otimes A_{k2} + \lambda_{a_2} \otimes A_{k3} + \lambda_{a_2^{-1}} \otimes A_{k4} + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{k(2k-1)} + \lambda_{a_k^{-1}} \otimes A_{k(2k)}, \\
\alpha(\lambda_{a_k^{-1}}) &= \lambda_{a_k} \otimes A_{k2}^* + \lambda_{a_1^{-1}} \otimes A_{k1}^* + \lambda_{a_2} \otimes A_{k4}^* + \lambda_{a_2^{-1}} \otimes A_{k3}^* + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{k(2k)}^* + \lambda_{a_k^{-1}} \otimes A_{k(2k-1)}^*.
\end{aligned}$$

From this we get the unitary corepresentation

$$U \equiv ((u_{ij})) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2k-1)} & A_{1(2k)} \\ A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \cdots & A_{1(2k)}^* & A_{1(2k-1)}^* \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2k-1)} & A_{2(2k)} \\ A_{22}^* & A_{21}^* & A_{24}^* & A_{23}^* & \cdots & A_{2(2k)}^* & A_{2(2k-1)}^* \\ \vdots & & & \vdots & & & \\ A_{k1} & A_{k2} & A_{k3} & A_{k4} & \cdots & A_{k(2k-1)} & A_{k(2k)} \\ A_{k2}^* & A_{k1}^* & A_{k4}^* & A_{k3}^* & \cdots & A_{k(2k)}^* & A_{k(2k-1)}^* \end{pmatrix}. \quad (1)$$

From now on, we call it as fundamental unitary. The coefficients A_{ij} and A_{ij}^* 's generate a norm-dense subalgebra of $\mathbb{Q}(\Gamma, S)$. We also note that the antipode of $\mathbb{Q}(\Gamma, S)$ maps u_{ij} to u_{ji}^* .

Remark 2.17 Using Corollary 2.15 and Proposition 2.16, $\mathbb{Q}(\Gamma, S)$ is the universal unital C^* -algebra generated by A_{ij} as above subject to the relations that U, U^t are unitaries and α given above is a C^* -homomorphism on $C_r^*(\Gamma)$.

Proposition 2.18 $(C^*(\Gamma), \Delta_\Gamma)$ always acts on $C_r^*(\Gamma)$ isometrically and faithfully, i.e. $(C^*(\Gamma), \Delta_\Gamma)$ is a subobject of $\mathbb{Q}(\Gamma, S)$ in the category $\hat{\mathbf{C}}(\mathbf{C}\Gamma, l^2(\Gamma), D_\Gamma)$.

Proof:

The usual (co)action of $C^*(\Gamma)$ on $C_r^*(\Gamma)$ coming from the coproduct of $C^*(\Gamma)$ gives us action on itself given by $\Delta_\Gamma(\lambda_g) = \lambda_g \otimes \lambda_g, \forall g \in \Gamma$. It is clear that

this action is isometric and faithful. So by Remark 2.17, there is a surjective C^* -morphism from $\mathbb{Q}(\Gamma)$ to $C^*(\Gamma)$ sending the entries $A_{i(2i-1)}$, $A_{i(2i-1)}^*$ of the fundamental unitary mentioned before to λ_{a_i} and $\lambda_{a_i^{-1}}$ respectively, others being sent to zero. The morphism intertwines the actions too. \square

The above proposition tells us that for a nonabelian group Γ , the quantum isometry group $\mathbb{Q}(\Gamma)$ is always a genuine CQG, i.e. the underlying C^* -algebra is non-commutative.

Corollary 2.19 $\mathbb{Q}(\Gamma, S) \cong (C^*(\Gamma), \Delta_\Gamma)$ if and only if the matrix (1) is diagonal.

proof:

Suppose $\mathbb{Q}(\Gamma, S) \cong (C^*(\Gamma), \Delta_\Gamma)$, then clearly (1) is a diagonal matrix. Conversely, let the matrix (1) is diagonal. By Proposition 2.18 we always get a surjective C^* -morphism from $\mathbb{Q}(\Gamma, S)$ to $(C^*(\Gamma), \Delta_\Gamma)$ sending $A_{i(2i-1)}$ to λ_{a_i} for each i . Now the action of $\mathbb{Q}(\Gamma, S)$ is defined as $\alpha(\lambda_{a_i}) = \lambda_{a_i} \otimes A_{i(2i-1)}$. So we get for $g = a_{i_1} a_{i_2} \cdots a_{i_k}$, $\alpha(\lambda_g) = \lambda_g \otimes A_g$ by the $*$ -homomorphism property of α , where $A_g \in \mathbb{Q}(\Gamma, S)$ is defined by $A_g := A_{i_1(2i_1-1)} \cdots A_{i_k(2i_k-1)}$. It is easy to see that the map $g \mapsto A_g$ satisfies $A_{gh} = A_g A_h$, $A_{g^{-1}} = A_g^*$, $A_e = 1$. Then by the universal property of $(C^*(\Gamma), \Delta_\Gamma)$ we get a surjective $*$ -morphism from $(C^*(\Gamma), \Delta_\Gamma)$ to $\mathbb{Q}(\Gamma, S)$ sending λ_{a_i} to $A_{i(2i-1)}$. This completes the proof. \square

We end the discussion of this subsection with the following easy observation which will be used in various places of this article.

Proposition 2.20 If $UV = 0$ for two normal elements in a C^* -algebra then

$$U^*V = VU^* = 0,$$

$$V^*U = UV^* = VU = 0.$$

2.3 Recollection of some known facts

To the best of our knowledge, first computations of $\mathbb{Q}(\Gamma)$ were done in [7]. Thereafter, several articles by different authors were devoted to computations of $\mathbb{Q}(\Gamma)$ for concrete groups. In [7] quantum isometry groups of cyclic groups (except \mathbb{Z}_4) were shown to be commutative. In case of \mathbb{Z}_4 it turns out to be noncommutative and infinite dimensional. It is in fact isomorphic with $C^*(D_\infty \times \mathbb{Z}_2)$ as a C^* -algebra (see [4]). Later in [2] it was identified with $\mathbb{Z}_2 \wr \mathbb{Z}_2$ as a quantum group.

The authors of [13] introduced the doubling procedure, and moreover they showed that for the symmetric group S_n with standard generating sets consisting of $(n-1)$ transpositions, the quantum isometry group coincides with the doubling of the group algebra. The same result holds for $D_{2(2n+1)}$ as well (see [22]). We will also briefly discuss the doubling procedure in Subsection 2.5.

In [3], Banica and Skalski introduced two parameter families $H_s^+(p, q)$, of quantum symmetry groups and they studied quantum isometry groups of duals of free product of cyclic groups for several cases in [4]. They showed that

$$\begin{aligned} H_n^+ &\cong \mathbb{Q}(\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 \cdots * \mathbb{Z}_2}_{n \text{ copies}}), \\ H^+(n, 0) &\cong \mathbb{Q}(\underbrace{\mathbb{Z} * \mathbb{Z} \cdots * \mathbb{Z}}_{n \text{ copies}}), \\ H_s^+(n, 0) &\cong \mathbb{Q}(\underbrace{\mathbb{Z}_s * \mathbb{Z}_s \cdots * \mathbb{Z}_s}_{n \text{ copies}}), \end{aligned}$$

where $s \neq 2, 4$. It was observed in [2]

$$C(O_n^{-1}) \cong \mathbb{Q}(\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \cdots \times \mathbb{Z}_2}_{n \text{ copies}}).$$

2.4 The case when Γ is a free or direct product

Theorem 2.21 *Let $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ be finitely generated discrete groups with the symmetric generating sets S_1, S_2, \dots, S_k respectively. Consider $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ with the generating set $S = \cup_{i=1}^k S'_i$, where $S'_i = (0, \dots, S_i, \dots, 0) \forall i = 1, \dots, k$. Then $\mathbb{Q}(\Gamma, S)$ has $\mathbb{Q}(\Gamma_1, S_1) \otimes^{\max} \mathbb{Q}(\Gamma_2, S_2) \otimes^{\max} \cdots \otimes^{\max} \mathbb{Q}(\Gamma_k, S_k)$ as a quantum subgroup.*

Proof:

Let $S_i = \{a_{ij}, j = 1, 2, \dots, k_i\}$ where $i = 1, 2, \dots, k$ and $((u_{pj}^{(i)}))$ be the fundamental unitary representation for the action of $\mathbb{Q}_i \equiv \mathbb{Q}(\Gamma_i, S_i)$ on $C_r^*(\Gamma_i)$. Consider the action $\alpha : C_r^*(\Gamma) \mapsto C_r^*(\Gamma) \hat{\otimes} \mathbb{Q}$ given by

$$\alpha(\lambda_{a'_{ip}}) = \sum_{j=1}^{k_i} \lambda_{a'_{ij}} \otimes 1_{(1)} \otimes 1_{(2)} \cdots \otimes 1_{(i-1)} \otimes u_{pj}^{(i)} \otimes 1_{(i+1)} \otimes \cdots \otimes 1_{(k)},$$

where $\mathbb{Q} = \mathbb{Q}_1 \otimes^{\max} \mathbb{Q}_2 \otimes^{\max} \cdots \otimes^{\max} \mathbb{Q}_k$ and $1_{(l)}$ denotes the identity element of the underlying C^* -algebra of \mathbb{Q}_l . It is easy to verify that this gives an isometric action of \mathbb{Q} on $C_r^*(\Gamma)$, hence by the universality of $\mathbb{Q}(\Gamma, S)$ we get a surjective morphism from $\mathbb{Q}(\Gamma, S)$ to \mathbb{Q} . \square

Remark 2.22 *In the set up of the previous theorem, replace the direct product by free product, i.e. take $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$. Then $\mathbb{Q}(\Gamma_1) \star \mathbb{Q}(\Gamma_2) \star \cdots \star \mathbb{Q}(\Gamma_k)$ is a quantum subgroup of $\mathbb{Q}(\Gamma)$. The proof is very similar to Theorem 2.21 and hence omitted.*

Remark 2.23 *We give an example to show that $\mathbb{Q}(H)$ may not be a quantum subgroup of $\mathbb{Q}(\Gamma)$ in general for a subgroup H of Γ , when Γ is neither $H \times K$ nor $H * K$ for some K . Take S_4 with generating sets (12), (23), (34) and H be the subgroup of it defined by $\langle (12), (34) \rangle$. H is clearly isomorphic with $\mathbb{Z}_2 \times \mathbb{Z}_2$. We know that $\mathbb{Q}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is infinite dimensional (see [2]) but the underlying C^* -algebra of $\mathbb{Q}(S_4)$ is isomorphic to $C^*(S_4) \oplus C^*(S_4)$ (result in [13]).*

Remark 2.24 We'll be usually interested to see whether $\mathbb{Q}(\Gamma_1) \star \mathbb{Q}(\Gamma_2) \star \cdots \star \mathbb{Q}(\Gamma_k)$ or $\mathbb{Q}(\Gamma_1) \otimes^{\max} \mathbb{Q}(\Gamma_2) \otimes^{\max} \cdots \otimes^{\max} \mathbb{Q}(\Gamma_k)$ coincides with $\mathbb{Q}(\Gamma)$ (in section 3 and 4) whenever Γ is either $\Gamma_1 \star \Gamma_2 \star \cdots \star \Gamma_k$ or $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ respectively. In this context the following observation will be useful.

Lemma 2.25 Let $((u_{ij}))$ be the fundamental representation of $\mathbb{Q}(\Gamma)$ on $C_r^*(\Gamma)$. Also assume $\Gamma = \Gamma_1 \star \Gamma_2 \star \cdots \star \Gamma_k$. Then $\mathbb{Q}(\Gamma) \cong \mathbb{Q}(\Gamma_1) \star \mathbb{Q}(\Gamma_2) \star \cdots \star \mathbb{Q}(\Gamma_k)$ if $((u_{ij}))$ is of the block diagonal form

$$U = \begin{pmatrix} C_1 & 0 \cdots & 0 \\ 0 & C_2 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & C_k \end{pmatrix}$$

with respect to the decomposition of the generating set S of Γ into $S_1 \cup S_2 \cup \cdots \cup S_k$.

Proof:

Let us write $S_i = \{a_{ij}, j = 1, 2, \dots, k_i\}$ as in Theorem 2.21. It is clear from the above form of U that for each i , the action (say α) of $\mathbb{Q}(\Gamma)$ maps $C_r^*(\Gamma_i) \cong C^*(\lambda_{a_{ij}}, j = 1, 2, \dots, k_i)$ to $C_r^*(\Gamma_i) \hat{\otimes} \mathbb{Q}(\Gamma, S)$ with the corresponding fundamental unitary being $C_i = ((c_{pj}^{(i)}))$. This means we have a morphism from $\mathbb{Q}_i \equiv \mathbb{Q}(\Gamma_i, S_i)$ to $\mathbb{Q}(\Gamma, S)$ sending $u_{pj}^{(i)}$ to $c_{pj}^{(i)}$. By definition of the free product, this gives a morphism from $\mathbb{Q}_1 \star \mathbb{Q}_2 \star \cdots \star \mathbb{Q}_k$ to $\mathbb{Q}(\Gamma)$. By Remark 2.22 we always get a surjective morphism from $\mathbb{Q}(\Gamma)$ to $\mathbb{Q}_1 \star \mathbb{Q}_2 \star \cdots \star \mathbb{Q}_k$ sending $c_{pj}^{(i)}$ to $u_{pj}^{(i)} \forall i$. This completes the proof. \square

Remark 2.26 The above lemma is true if we replace the free product by direct product, i.e. $\mathbb{Q}(\Gamma) \cong \mathbb{Q}(\Gamma_1) \otimes^{\max} \mathbb{Q}(\Gamma_2) \otimes^{\max} \cdots \otimes^{\max} \mathbb{Q}(\Gamma_k)$ if and only if $((u_{ij}))$ is of the block diagonal form. Moreover, entries of one such block commute with the entries of any other block. The proof of this fact is very similar to Lemma 2.25, hence omitted.

2.5 $\mathbb{Q}(\Gamma)$ as a doubling of $C^*(\Gamma)$

We briefly recall the doubling procedure of the group algebra from [13], [21]. Let (\mathcal{Q}, Δ) be a CQG with a CQG-automorphism θ such that $\theta^2 = id$. The doubling of this CQG, say $(\mathcal{D}_\theta(\mathcal{Q}), \tilde{\Delta})$, is given by $\mathcal{D}_\theta(\mathcal{Q}) := \mathcal{Q} \oplus \mathcal{Q}$ (direct sum as a C^* -algebra), and the coproduct is defined by the following, where we have denoted the injections of \mathcal{Q} onto the first and second coordinate in $\mathcal{D}_\theta(\mathcal{Q})$ by ξ and η respectively, i.e. $\xi(a) = (a, 0)$, $\eta(a) = (0, a)$, $(a \in \mathcal{Q})$.

$$\tilde{\Delta} \circ \xi = (\xi \otimes \xi + \eta \otimes [\eta \circ \theta]) \circ \Delta,$$

$$\tilde{\Delta} \circ \eta = (\xi \otimes \eta + \eta \otimes [\xi \circ \theta]) \circ \Delta.$$

Below we give a sufficient condition for the quantum isometry group to be a doubling of the group algebra. For this, it is convenient to use a slightly

different notational convention: let $U_{2i-1,j} = A_{ij}$ for $i = 1, \dots, k$, $j = 1, \dots, 2k$ and $U_{2i,2l} = A_{i(2l-1)}^*$, $U_{2i,2l-1} = A_{i(2l)}^*$ for $i = 1, \dots, k$, $l = 1, \dots, k$.

Lemma 2.27 *Let Γ be a group with k generators $\{a_1, a_2, \dots, a_k\}$ and define $\gamma_{2l-1} := a_l$, $\gamma_{2l} := a_l^{-1} \forall l = 1, 2, \dots, k$. Now σ be an order 2 automorphism on the set $\{1, 2, \dots, 2k-1, 2k\}$ and θ be an automorphism of the group given by $\theta(\gamma_i) = \gamma_{\sigma(i)}$. We assume the following:*

1. $B_i := U_{i,\sigma(i)} \neq 0 \forall i$ and $U_{i,j} = 0 \forall j \notin \{\sigma(i), i\}$,
2. $A_i B_j = B_j A_i = 0 \forall i, j$ such that $\sigma(i) \neq i$ and $\sigma(j) \neq j$, where $A_i = U_{i,i}$,
3. All $U_{i,j} U_{i,j}^*$ are central projections,
4. There are well defined C^* -isomorphisms π_1, π_2 from $C^*(\Gamma)$ to $C^*\{A_i, i = 1, 2, \dots, 2k\}$ and $C^*\{B_i, i = 1, 2, \dots, 2k\}$ respectively such that

$$\pi_1(\lambda_{a_i}) = A_i, \quad \pi_2(\lambda_{a_i}) = B_i \forall i.$$

Then $\mathbb{Q}(\Gamma)$ is doubling of $(C^*(\Gamma, \Delta_\Gamma)$ corresponding to a given automorphism θ . Moreover, the fundamental unitary takes the following form

$$\begin{pmatrix} A_1 & 0 & 0 & 0 & \cdots & 0 & B_1 \\ 0 & A_2 & 0 & 0 & \cdots & B_2 & 0 \\ 0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & A_4 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \\ 0 & B_{2k-1} & 0 & 0 & \cdots & A_{2k-1} & 0 \\ B_{2k} & 0 & 0 & 0 & \cdots & 0 & A_{2k} \end{pmatrix}.$$

Proof:

First note that when $\sigma(i) = i$, then i -th row contains only one non-zero element A_i , then $U_{i,i} := A_i = B_i$. Without loss of generality we assume that there are first k_1 number of i 's from the set $\{1, 2, \dots, 2k-1, 2k\}$ such that $\sigma(i) \neq i$ and k_2 number of j 's from the set $\{k_1 + 1, \dots, 2k\}$ such that $\sigma(j) = j$. This implies $k_1 + k_2 = 2k$ and always $k_1 \geq 1$ as σ is non-trivial. Now the C^* -algebra $C^*(\Gamma) \oplus C^*(\Gamma)$ is generated by $(\lambda_{\gamma_i} \oplus 0), (0 \oplus \lambda_{\gamma_{\sigma(i)}}), i = 1, 2, \dots, 2k$. We can define a C^* -homomorphism π from $\mathcal{D}_\theta(C^*(\Gamma))$ to $\mathbb{Q}(\Gamma)$ given by

$$\pi(\lambda_{\gamma_i} \oplus 0) = A_i, \quad \pi(0 \oplus \lambda_{\gamma_{\sigma(i)}}) = B_i \forall i = 1, 2, \dots, k_1,$$

$$\pi(\lambda_{\gamma_j} \oplus 0) = A_j A_1 A_1^*, \quad \pi(0 \oplus \lambda_{\gamma_{\sigma(j)}}) = A_j B_1 B_1^* \forall j = k_1 + 1, \dots, 2k.$$

It is easy to verify that this is indeed a CQG isomorphism. \square

2.6 $\mathbb{Q}(\Gamma, S)$ as a quantum isometry group of metric space

In this subsection, our aim is to identify $\mathbb{Q}(\Gamma, S)$ with the quantum isometry group of some metric space in the sense of [15], in case Γ is abelian. We first recall the definition of quantum isometry group in the purely metric space setting. Given a compact metric space (X, d) , we say that an action α of a CQG \mathbb{Q} on $C(X)$ is isometric if the action $\alpha_r = (id \otimes \pi_r) \circ \alpha$ of the reduced CQG \mathbb{Q}_r (where $\pi_r : \mathbb{Q} \rightarrow \mathbb{Q}_r$ is the canonical map from \mathbb{Q} to the reduced CQG \mathbb{Q}_r) satisfies

$$\alpha_r(d_x)(y) = \kappa(\alpha_r(d_y)(x)),$$

$\forall x, y \in X$, where $d_x(\cdot) \equiv d(x, \cdot)$ and κ denotes the (norm bounded by Theorem 3.23 in [16]) antipode of \mathbb{Q}_r . It is shown in [15] that in case $X \subseteq \mathbb{R}^n$ isometrically embedded, the above condition is equivalent to the following:

$$\sum_i (F_i(x) - F_i(y))^2 = \sum_i (x_i - y_i)^2 \cdot 1,$$

where $F_i(x) = \alpha(X_i)(x)$ and X_i denotes the i -th coordinate function of \mathbb{R}^n , restricted to X . It can be easily seen, by almost verbatim adaptation of the arguments in [15], that a similar result would hold if we replaced \mathbb{R}^n by \mathbb{C}^n . That is, for $X \subseteq \mathbb{C}^n$ isometrically, with the metric $d(z, w)^2 = \sum_i |z_i - w_i|^2$, a CQG action $\alpha : C(X) \rightarrow C(X) \hat{\otimes} \mathbb{Q}$ is isometric in the metric space sense if and only if

$$\sum_i (F_i(z) - F_i(w))^* (F_i(z) - F_i(w)) = d^2(z, w) \cdot 1,$$

$\forall z, w \in X$. Moreover, it is clear that a sufficient condition for the above is that

$$\sum_i F_i(z)^* F_i(w) = \langle z, w \rangle \cdot 1 \equiv \sum_i \bar{z}_i w_i \cdot 1.$$

We can also prove, as in [15], that for metric spaces (X, d) isometrically embedded in \mathbb{C}^n , there exists a universal CQG acting isometrically on it, to be denoted by $\text{QISO}(X, d)$, and its action is affine in the sense that $\alpha(Z_i) = \sum_j Z_j \otimes b_{ij} + 1 \otimes R_i$ for some b_{ij} and R_i , where Z_i is the coordinate function restricted to $X \subseteq \mathbb{C}^n$.

Let us now consider a finitely generated abelian group Γ with a symmetric generating set $S = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ and let $((q_{ij}))$ be the fundamental unitary for $\mathbb{Q}(\Gamma, S)$. Now consider the dual group of Γ say $G = \hat{\Gamma}$, which is a compact topological group. Moreover, $\hat{\Gamma}$ can be identified with a compact subset X of \mathbb{C}^n via the map $\chi \mapsto (\chi(\gamma_1), \chi(\gamma_2), \dots, \chi(\gamma_n))$. There is a natural Euclidean metric \hat{d}_S on $\hat{\Gamma}$ given by $\hat{d}_S(\chi, \chi')^2 = \sum_i |\chi(\gamma_i) - \chi'(\gamma_i)|^2$.

Theorem 2.28 *With the above set-up, we have*

$$\mathbb{Q}(\Gamma, S) \cong \text{QISO}(\hat{\Gamma}, \hat{d}_S).$$

Proof:

It is well-known that $C_r^*(\Gamma)$ is isomorphic with $C(\hat{\Gamma})$ via the Fourier transform \mathcal{F} , extended as a unitary from $\ell^2(\Gamma)$ to $L^2(\hat{\Gamma})$ i.e. $C(\hat{\Gamma}) = \mathcal{F}C_r^*(\Gamma)\mathcal{F}^{-1} \subseteq B(L^2(\hat{\Gamma}))$. Let U denote the unitary representation of $\mathbb{Q}(\Gamma, S)$ on $C_r^*(\Gamma)$ and let U' and α' be the corresponding representation and action of $\mathbb{Q}(\Gamma, S)$ on $C(\hat{\Gamma})$ respectively, i.e. $U' = \mathcal{F}U\mathcal{F}^{-1}$ and $\alpha'(Z_i) = (\mathcal{F} \otimes 1)\alpha(Z_i)(\mathcal{F}^{-1} \otimes 1)$. As $\lambda_{\gamma_1}, \lambda_{\gamma_2}, \dots, \lambda_{\gamma_n}$ generate $C_r^*(\Gamma)$ as a C^* -algebra, under the Fourier transform $C(\hat{\Gamma})$ is generated by $\hat{\lambda}_{\gamma_1}, \hat{\lambda}_{\gamma_2}, \dots, \hat{\lambda}_{\gamma_n}$. We have $Z_i = \hat{\lambda}_{\gamma_i}$ as the coordinate functions restricted to $\hat{\Gamma} \subset \mathbb{C}^n$. Clearly, $C(\hat{\Gamma}) \cong C^*\{Z_1, Z_2, \dots, Z_n\} \subseteq C_0(\mathbb{C}^n)$. Let $F(z) = (F_1(z), F_2(z), \dots, F_n(z))$ for $z \in \hat{\Gamma}$, where $F_i(z) := \alpha'(Z_i)(z)$. Now it is clear that $F_i \equiv \alpha'(Z_i) = \sum_j Z_j \otimes q_{ij}$, and as $((q_{ij}))$ is unitary, it is easy to verify that $\sum_i F_i(z)^* F_i(w) = \sum_i \bar{z}_i w_i \otimes 1_{\mathbb{Q}}$. Thus α' is an isometric action of $\mathbb{Q}(\Gamma, S)$ on $C(\hat{\Gamma})$, i.e. $\mathbb{Q}(\Gamma, S)$ is a quantum subgroup of $\text{QISO}(\hat{\Gamma}, \hat{d}_S)$ using the surjective morphism $b_{ij} \mapsto q_{ij}, R_i \mapsto 0$.

Conversely, assume that the isometric action of $\mathbb{Q}' \equiv \text{QISO}(\hat{\Gamma}, \hat{d}_S)$ is given by $\beta(Z_i) = \sum_j Z_j \otimes b_{ij} + 1 \otimes R_i$ for some b_{ij} and R_i . Consider any Borel, probability measure on $(\hat{\Gamma}, \hat{d}_S)$ and convolving with the Haar state of \mathbb{Q}' , we get a \mathbb{Q}' invariant probability measure say μ on $\hat{\Gamma}$. As $\hat{\Gamma}$ itself acts isometrically on $(\hat{\Gamma}, \hat{d}_S)$, $C_r^*(\Gamma) (\cong C(\hat{\Gamma}))$ is a quantum subgroup of \mathbb{Q}' , hence μ is a $\hat{\Gamma}$ invariant probability measure on $\hat{\Gamma}$. Therefore μ must be the (unique) Haar measure of $\hat{\Gamma}$. As the state corresponding to the Haar measure (say ϕ) maps each Z_i to zero, we conclude using the β invariance of ϕ i.e. $(\phi \otimes id)\beta(Z_i) = \phi(Z_i)1_{\mathbb{Q}'}$, that $R_i = 0$ for each i . Thus β is linear, i.e. $\beta(Z_i) = \sum_j Z_j \otimes b_{ij}$. The corresponding $*$ -homomorphism from $C_r^*(\Gamma)$ to $C_r^*(\Gamma) \hat{\otimes} \mathbb{Q}'$ maps λ_{γ_i} to $\sum_j \lambda_{\gamma_j} \otimes b_{ij}$ and it is clearly an action. Finally, as β preserves the Haar state μ , it extends to a unitary representation on the L^2 -space, hence in particular, $((b_{ij}))$ is the matrix corresponding to a unitary representation. In other words, $((b_{ij}))$ is a unitary element of $M_n(\mathbb{Q}')$. This shows by the definition of $\mathbb{Q}(\Gamma, S)$ that \mathbb{Q}' is a quantum subgroup of $\mathbb{Q}(\Gamma, S)$, with the surjective morphism sending q_{ij} to b_{ij} . This completes the proof. \square

2.7 Polynomial growth of $\mathbb{Q}(\Gamma)$

We briefly discuss some sufficient conditions for the quantum group $\mathbb{Q}(\Gamma)$ to have polynomial growth property in the sense of [5], when Γ has polynomial growth.

We now state and prove the main result of this subsection.

Theorem 2.29 *Let Γ be a finitely generated discrete group with polynomial growth and assume that the action of $\mathbb{Q}(\Gamma)$ on $C_r^*(\Gamma)$ has full spectrum. Then the dual discrete quantum group of $\mathbb{Q}(\Gamma)$ also has polynomial growth property.*

Proof:

Let S be a finite generating set for Γ . As the group Γ has polynomial growth, there is some polynomial p of one variable such that the cardinality of the set

$\{g_1 g_2 \dots g_m : g_i \in S \ \forall i, m \leq n\}$ is bounded by $p(n)$ for each n . That is, the dimension of the vector space, say \mathcal{V}_n , spanned by elements of the form $\lambda_{a_1} \dots \lambda_{a_m}$ where $a_i \in S$ and $m \leq n$, has dimension less than or equal to $p(n)$. But this space is clearly left invariant by the action of $\mathbb{Q}(\Gamma)$. Let us denote the restriction of this action to \mathcal{V}_n , which is a finite dimensional unitary representation, by π_n . Moreover, by the assumption of full spectrum, every irreducible representation of $\mathbb{Q}(\Gamma)$ must be a sub-representation of some π_n for sufficiently large n . This allows us to define a central length function l on the set of irreducible representation of $\mathbb{Q}(\Gamma)$ (in the sense of [5]) by setting $l(\pi)$ to be equal to the smallest value of n for which π is a sub-representation of π_n . Clearly,

$$\sum_{\pi: l(\pi) \leq n} d_\pi^2 \leq p(n),$$

where d_π denotes the dimension of the irreducible π . From this, it is easily seen that this length function satisfies the criteria of Definition 4.1 of [5], hence the dual of \mathbb{Q} has polynomial growth. \square

2.8 The structure of the maximal commutative subgroup of $\mathbb{Q}(\Gamma)$ for $\Gamma = \mathbb{Z}_n^k$

Proposition 2.30 *Let $\Gamma = \underbrace{(\mathbb{Z}_n \times \mathbb{Z}_n \times \dots \times \mathbb{Z}_n)}_{k \text{ copies}}$ and S_k be the group of permutation of k elements. We have:*

1. *If $n = 2$ then we have $C(ISO(\Gamma)) \cong C(\underbrace{(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2)}_{k \text{ copies}} \rtimes S_k)$.*
2. *If $n \neq 2, 4$ then $C(ISO(\Gamma)) \cong C(\underbrace{(\mathbb{Z}_n \times \mathbb{Z}_n \times \dots \times \mathbb{Z}_n)}_{k \text{ copies}} \rtimes (\mathbb{Z}_2^k \rtimes S_k))$.*

Proof:

Let a_1, a_2, \dots, a_k be the usual generating elements for Γ . The fundamental unitary is of the form

$$U \equiv ((u_{ij})) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \dots & A_{1(2k-1)} & A_{1(2k)} \\ A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \dots & A_{1(2k)}^* & A_{1(2k-1)}^* \\ A_{21} & A_{22} & A_{23} & A_{24} & \dots & A_{2(2k-1)} & A_{2(2k)} \\ A_{22}^* & A_{21}^* & A_{24}^* & A_{23}^* & \dots & A_{2(2k)}^* & A_{2(2k-1)}^* \\ \vdots & & & \vdots & & & \\ A_{k1} & A_{k2} & A_{k3} & A_{k4} & \dots & A_{k(2k-1)} & A_{k(2k)} \\ A_{k2}^* & A_{k1}^* & A_{k4}^* & A_{k3}^* & \dots & A_{k(2k)}^* & A_{k(2k-1)}^* \end{pmatrix}.$$

However, we have the additional condition that u_{ij} 's commute among themselves and each of them is normal. Our first claim is $A_{ij} A_{ik} = 0 = A_{ji} A_{ki}, \forall i, j, k$ with $j \neq k$.

We break it into two cases.

Case 1: $n = 2$: Consider the term $\alpha(\lambda_{a_i^2}) = \lambda_e \otimes 1_{\mathbb{Q}} \forall i$. Now comparing the coefficients of $\lambda_{a_l a_m} \forall l \neq m$ on both sides of the equation we obtain $A_{il}A_{im} = 0 \forall l \neq m$. Applying the antipode we find $A_{li}A_{mi} = 0 \forall l \neq m$.

Case 2: $n \neq 2, 4$: Using the relation $\alpha(\lambda_{a_j})\alpha(\lambda_{a_j^{-1}}) = \alpha(\lambda_{a_j^{-1}})\alpha(\lambda_{a_j}) = \alpha(\lambda_e) = \lambda_e \otimes 1_{\mathbb{Q}}$ and comparing the coefficients of $\lambda_{a_i^2}$ and $\lambda_{a_i^{-2}}$ on both sides we must have

$$A_{i(2j-1)}A_{i(2j)}^* = A_{i(2j)}^*A_{i(2j-1)} = 0. \quad (2)$$

Applying the antipode on (2) we get that $A_{i(2j-1)}^*A_{i(2j)}^* = A_{i(2j)}^*A_{i(2j-1)}^* = 0$, which shows

$$A_{i(2j)}A_{i(2j-1)} = A_{i(2j-1)}A_{i(2j)} = 0 \forall i, j. \quad (3)$$

Now, we will show that $A_{i(2j)}A_{im} = A_{i(2j-1)}A_{im} = 0$, where $m \neq 2j, (2j-1)$. Using the condition $\alpha(\lambda_{a_i})\alpha(\lambda_{a_i^{-1}}) = \alpha(\lambda_{a_i^{-1}})\alpha(\lambda_{a_i}) = \alpha(\lambda_e) = \lambda_e \otimes 1_{\mathbb{Q}}$ and comparing the coefficients of $\lambda_{a_j a_l}, \lambda_{a_j a_l^{-1}}$ where $j \neq l$ one can get

$$A_{i(2j-1)}A_{i(2l)}^* + A_{i(2l-1)}A_{i(2j)}^* = 0, \quad (4)$$

$$A_{i(2j)}A_{i(2l-1)}^* + A_{i(2l)}A_{i(2j-1)}^* = 0. \quad (5)$$

Multiplying $A_{i(2j-1)}^*$ and $A_{i(2j)}^*$ on the right side of the equations (4) and (5) respectively we have $A_{i(2j-1)}A_{i(2l)}^*A_{i(2j-1)}^* = A_{i(2j)}A_{i(2l-1)}^*A_{i(2j)}^* = 0$. Thus, $A_{i(2j-1)}A_{i(2l)}^* = A_{i(2j)}A_{i(2l-1)}^* = 0$ by using the C^* -norm condition and commutativity among A_{ij} 's. Now using the Proposition 2.20 we get $A_{i(2j-1)}A_{i(2l)} = A_{i(2j)}A_{i(2l-1)} = 0$. Similarly, we have $A_{i(2j)}A_{i(2l)} = A_{i(2j-1)}A_{i(2l-1)} = 0$. This proves the claim by using antipode. Moreover, for finite n , we have $A_{ij}^* = A_{ij}^{n-1}$.

Now we produce the explicit isomorphism. Identify $C(\underbrace{(\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n)}_{k \text{ copies}}) \rtimes (\mathbb{Z}_2^k \rtimes S_k)$ as a C^* -algebra in a natural way with $C(\underbrace{(\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n)}_{k \text{ copies}}) \otimes C(\mathbb{Z}_2^k) \otimes C(S_k)$.

Let $\chi_{ij} \in C(S_k)$ be the characteristic function of the set of those permutations which map i to j and also $\eta_{0,i}, \eta_{1,i} \in C(\mathbb{Z}_2^k)$ be the characteristic functions of sets which have respectively 0 or 1 in the i -th coordinate. One can easily check as in Theorem 6.1 of [3] that the map

$$A_{i(2j-1)} \mapsto z_i \otimes \chi_{ij} \otimes \eta_{0,i},$$

$$A_{i(2j)} \mapsto z_i \otimes \chi_{ij} \otimes \eta_{1,i},$$

gives an isomorphism between $C(ISO(\Gamma))$ and $C(\underbrace{(\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n)}_{k \text{ copies}}) \rtimes (\mathbb{Z}_2^k \rtimes S_k)$,

and that it preserves the respective coproducts. So it becomes a CQG isomorphism. \square

3 Computations for free product of cyclic groups

Theorem 3.1 Let $\Gamma = \Gamma_1 * \Gamma_2 \cdots * \Gamma_l$ where $\Gamma_i = \underbrace{(\mathbb{Z}_{n_i} * \mathbb{Z}_{n_i} \cdots * \mathbb{Z}_{n_i})}_{k_i \text{ copies}}$. Also

assume $n_1 \neq n_2 \neq \cdots \neq n_l$ and $n_i \neq 2, 4 \forall i$, then $\mathbb{Q}(\Gamma)$ will be $H_{n_1}^+(k_1, 0) \star H_{n_2}^+(k_2, 0) \star \cdots \star H_{n_l}^+(k_l, 0)$. i.e. $\mathbb{Q}(\Gamma) \cong \mathbb{Q}(\Gamma_1) \star \mathbb{Q}(\Gamma_2) \star \cdots \star \mathbb{Q}(\Gamma_l)$.

Proof:

For simplicity of notation we present the case when all $k_i = 1$. The general case will follow by essentially the same arguments, with just an extra bit of careful book-keeping of notations. So let $\Gamma = \mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \cdots * \mathbb{Z}_{n_l}$ and let a_i be the standard generators of \mathbb{Z}_{n_i} , $o(a_i) = n_i \forall i$. Take $\{a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_l, a_l^{-1}\}$ as the symmetric generating set for this group. Without loss of generality we assume $n_1 < n_2 < \cdots < n_l$. n_l can take the value ∞ also. Now the fundamental unitary of $\mathbb{Q}(\Gamma)$ is of the form

$$U = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2l-1)} & A_{1(2l)} \\ A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \cdots & A_{1(2l)}^* & A_{1(2l-1)}^* \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\ A_{22}^* & A_{21}^* & A_{24}^* & A_{23}^* & \cdots & A_{2(2l)}^* & A_{2(2l-1)}^* \\ \vdots & & & \vdots & & & \\ A_{l1} & A_{l2} & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\ A_{l2}^* & A_{l1}^* & A_{l4}^* & A_{l3}^* & \cdots & A_{l(2l)}^* & A_{l(2l-1)}^* \end{pmatrix}.$$

Our aim is to show that it reduces to the form

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\ A_{12}^* & A_{11}^* & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & \cdots & 0 & 0 \\ 0 & 0 & A_{24}^* & A_{23}^* & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & A_{l(2l-1)} & A_{l(2l)} \\ 0 & 0 & 0 & 0 & \cdots & A_{l(2l)}^* & A_{l(2l-1)}^* \end{pmatrix}$$

i.e. only the diagonal (2×2) block will survive and others become zero. By Lemma 2.25 this will complete the proof.

We break the proof into a number of lemmas.

Lemma 3.2 $A_{i(2j)} A_{i(2j-1)} = A_{i(2j-1)} A_{i(2j)} = 0 \forall i, j$.

Proof:

Using the relation $\alpha(\lambda_{a_j})\alpha(\lambda_{a_j^{-1}}) = \alpha(\lambda_{a_j^{-1}})\alpha(\lambda_{a_j}) = \alpha(\lambda_e) = \lambda_e \otimes 1_{\mathbb{Q}}$ and comparing the coefficients of $\lambda_{a_i^2}$ and $\lambda_{a_i^{-2}}$ on both sides we have $A_{i(2j-1)}A_{i(2j)}^* = A_{i(2j)}^*A_{i(2j-1)} = 0$. Applying the antipode one can get that $A_{i(2j-1)}^*A_{i(2j)} = A_{i(2j)}^*A_{i(2j-1)}^* = 0$, thus $A_{i(2j)}A_{i(2j-1)} = A_{i(2j-1)}A_{i(2j)} = 0$. \square

Lemma 3.3 $A_{1j} = 0 \ \forall \ j > 2$.

Proof:

First we fix some notational convention for clarity of exposition. Let $b_{2i-1} = a_i$ and $b_{2i} = a_i^{-1}$, $i = 1, 2, \dots, l$. Now consider $\alpha(\lambda_{a_1^{n_1-1}}) = \alpha(\lambda_{a_1^{-1}})$. Observe that both sides must contain only the words of length 1, i.e. only b_i 's occur in the chain. For $\underline{i} = (i_1, i_2, \dots, i_m)$, set $A_{\underline{i}} \equiv A_{1i_1}A_{1i_2} \cdots A_{1i_m}$, $b_{\underline{i}} \equiv b_{i_1}b_{i_2} \cdots b_{i_m}$ and $\Lambda = \{(i_1, i_2, \dots, i_{n_1-1}) \mid i_j \in \{1, 2, \dots, 2l\}\}$, note that L.H.S. $= \sum_{\underline{i} \in \Lambda} \lambda_{b_{\underline{i}}} \otimes A_{\underline{i}}$. Using Lemma 3.2 we conclude that $A_{\underline{i}} = 0$ whenever $b_{\underline{i}}$ is not a reduced word, i.e. there is some i_r such that $b_{i_r}^{-1} = b_{i_{r+1}}$. Thus the L.H.S. reduces to $\sum_{\underline{i} \in \Lambda} \lambda_{b_{\underline{i}}} \otimes A_{\underline{i}}$, where $b_{\underline{i}}$ is a reduced word. Our claim is that there are no coefficients of λ_{a_d} and $\lambda_{a_d^{-1}} \ \forall \ d > 1$ in L.H.S. Suppose $a_{j_1}^{m_1}a_{j_2}^{m_2} \cdots a_{j_p}^{m_p} = a_2$ where $j_i \neq j_{i+1} \ \forall \ i$ and $|m_1| + |m_2| + \cdots + |m_p| = n_1 - 1$ which implies $a_{j_1}^{m_1}a_{j_2}^{m_2} \cdots a_{j_p}^{m_p}a_2^{-1} = e$. This shows that p must be 1 and $j_1 = 2$. Hence we can write $a_2^{m_1} = a_2$, $|m_1| = n_1 - 1$, leading to a contradiction as $n_1 < n_2$. We get similar contradiction if we replace a_2 by any other $a_i \ \forall \ i > 2$ as $n_1 < n_i \ \forall \ i > 2$. Thus comparing the coefficients of λ_{a_d} and $\lambda_{a_d^{-1}} \ \forall \ d > 1$ on both sides one can deduce $A_{1j}^* = 0 \ \forall \ j > 2$. Taking the adjoint we get the desired result. \square

Now applying the antipode we find that $A_{i1} = A_{i2} = 0 \ \forall \ i > 1$.

The structure of the unitary matrix reduces to the form

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\ A_{12}^* & A_{11}^* & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\ 0 & 0 & A_{24}^* & A_{23}^* & \cdots & A_{2(2l)}^* & A_{2(2l-1)}^* \\ \vdots & & & & & & \\ 0 & 0 & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\ 0 & 0 & A_{l4}^* & A_{l3}^* & \cdots & A_{l(2l)}^* & A_{l(2l-1)}^* \end{pmatrix}.$$

If we repeat this line of reasoning with other generators starting with a_2 and so on, we get the desired block diagonal form and hence complete the proof. One can easily observe that the last condition $\alpha(\lambda_{a_1^{n_1-1}}) = \alpha(\lambda_{a_1^{-1}})$ is not used in the proof. So we can include the $n_l = \infty$ case also. \square

We'll now show how to extend Theorem 3.1 in some cases when 2 or 4 can occur in n_1, n_2, \dots, n_l .

Theorem 3.4 *Theorem 3.1 is valid also for the case when $n_1 = 2$ and $2 \neq n_2 \neq \dots \neq n_l$ where $n_i \neq 4, \infty \forall i$.*

Proof:

As in the proof of Theorem 3.1 we assume $k_i = 1 \forall i$ for simplicity of exposition and moreover $n_2 < n_3 < \dots < n_l$. From the relation $\alpha(\lambda_{a_1^2}) = \lambda_e \otimes 1_{\mathbb{Q}}$, comparing the coefficients of $\lambda_{a_2^2}, \lambda_{a_2^{-2}}, \lambda_{a_3^2}, \lambda_{a_3^{-2}}, \dots, \lambda_{a_l^2}, \lambda_{a_l^{-2}}$ on both sides we find that

$$A_{12}^2 = A_{13}^2 = \dots = A_{1(2l-2)}^2 = A_{1(2l-1)}^2 = 0. \quad (6)$$

This implies

$$A_{21}^2 = (A_{21}^*)^2 = \dots = A_{k1}^2 = (A_{k1}^*)^2 = 0 \quad (7)$$

by applying the antipode on (6). Also observe that

$$A_{2(2j+1)} A_{2(2j)} = A_{2(2j)} A_{2(2j+1)} = 0 \forall j \quad (8)$$

by the similar argument of Lemma 3.2.

Now with the notations of Lemma 3.3, consider $\alpha(\lambda_{a_2^{n_2-1}}) = \alpha(\lambda_{a_2^{-1}})$. L.H.S contains terms of the form $\sum_{\underline{i} \in \Lambda} \lambda_{b_{\underline{i}}} \otimes A_{\underline{i}}$ where $\Lambda = \{(i_1, i_2, \dots, i_{n_2-1}) \mid i_j \in \{1, 2, \dots, (2l-1)\}\}$. Note that here also $b_{\underline{i}}$ is reduced word by the equations (7), (8) and the fact $n_2 < n_3 < \dots < n_l$. Thus comparing the coefficients of $\lambda_{a_1}, \lambda_{a_p}, \lambda_{a_p^{-1}} \forall p > 2$ on both sides of the relation $\alpha(\lambda_{a_2^{n_2-1}}) = \alpha(\lambda_{a_2^{-1}})$ one can deduce that $A_{21}^* = A_{2k}^* = 0 \forall k > 3$. Applying the antipode we have, $A_{12} = A_{13} = 0$ and $A_{k2} = A_{k3} = 0 \forall k > 2$.

So the fundamental Unitary is reduced to the form

$$\begin{pmatrix} A_{11} & 0 & 0 & \dots & A_{1(2l-2)} & A_{1(2l-1)} \\ 0 & A_{22} & A_{23} & \dots & 0 & 0 \\ 0 & A_{23}^* & A_{22}^* & \dots & 0 & 0 \\ \vdots & & & \ddots & & \\ A_{l1} & 0 & 0 & \dots & A_{l(2l-2)} & A_{l(2l-1)} \\ A_{l1}^* & 0 & 0 & \dots & A_{l(2l-1)}^* & A_{l(2l-2)}^* \end{pmatrix}.$$

Repeating similar arguments with a_3, a_4, \dots, a_l we will get the desired block diagonal form and complete the proof. \square

Remark 3.5 *We see that finiteness of order of a_l is used in the last paragraph of Theorem 3.4. But in Theorem 2.1 this condition is not used. In fact it won't be necessary for any of the results that follow. So this is the only case where the finiteness condition is necessary.*

Theorem 3.6 *The result of Theorem 3.1 remains true for $n_1 = 4, 4 \neq n_2 \neq \dots \neq n_l$ and $n_i \neq 2 \forall i$.*

Proof:

We continue using the notation and convention of Theorem 3.1 and without loss of generality assume $k_i = 1 \forall i$ and $o(a_1) = 4$. First consider the term $\alpha(\lambda_{a_1^2})$ and note that the coefficient of λ_e in the expression of $\alpha(\lambda_{a_1^2})$ must be zero. Thus we have

$$A_{12}A_{11} + A_{11}A_{12} + \cdots + A_{1(2l-1)}A_{1(2l)} + A_{1(2l)}A_{1(2l-1)} = 0. \quad (9)$$

Now, observe that

$$A_{1(2i-1)}A_{1(2i)} = A_{1(2i)}A_{1(2i-1)} = 0 \forall i > 1 \quad (10)$$

by the same argument of Lemma 3.2. From (9) and (10) we have

$$A_{12}A_{11} + A_{11}A_{12} = 0. \quad (11)$$

Next, we want to show that $A_{1i} = 0 \forall i > 2$. From the equation $\alpha(\lambda_{a_1^3}) = \alpha(\lambda_{a_1^{-1}})$ comparing the coefficients of $\lambda_{a_2}, \lambda_{a_2^{-1}}, \lambda_{a_3}, \lambda_{a_3^{-1}}, \dots, \lambda_{a_l}, \lambda_{a_l^{-1}}$ on both sides we deduce

$$A_{1(2i)}^* = A_{1(2i-1)}(A_{12}A_{11} + A_{11}A_{12} + \cdots + A_{1(2i-1)}A_{1(2i)} + \cdots + A_{1(2l)}A_{1(2l-1)}),$$

$$A_{1(2i-1)}^* = A_{1(2i)}(A_{12}A_{11} + A_{11}A_{12} + \cdots + A_{1(2i)}A_{1(2i-1)} + \cdots + A_{1(2l)}A_{1(2l-1)}),$$

where $\forall i > 1$.

Now, using (10) and (11) one can conclude that $A_{1i} = 0 \forall i > 2$. The fundamental unitary is of the form

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\ A_{12}^* & A_{11}^* & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\ 0 & 0 & A_{24}^* & A_{23}^* & \cdots & A_{2(2l)}^* & A_{2(2l-1)}^* \\ \vdots & & & \vdots & & & \\ 0 & 0 & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\ 0 & 0 & A_{l4}^* & A_{l3}^* & \cdots & A_{l(2l)}^* & A_{l(2l-1)}^* \end{pmatrix}.$$

We then follow the strategy of Theorem 3.1 to complete the proof. \square

Theorem 3.7 *The conclusion of Theorem 3.1 holds also for $n_1 = 2, n_2 = 4$ where $n_3 \neq n_4 \neq \cdots \neq n_l$ and $n_i \neq 2, 4, \infty \forall i \geq 3$.*

Proof:

With the notation and convention as before, we get from the condition $\alpha(\lambda_{a_2^3}) = \alpha(\lambda_{a_2^{-1}})$ comparing the coefficient of λ_{a_1} on both sides

$$A_{21}^* = A_{21}(A_{22}A_{23} + A_{23}A_{22}) \quad (12)$$

as $A_{2(2i+1)}A_{2(2i)} = A_{2(2i)}A_{2(2i+1)} = 0 \ \forall \ i > 1$ by arguments similar to those of Lemma 3.2.

Again, considering the coefficient of λ_e in the expression of $\alpha(\lambda_{a_2^2})$ we have

$$(A_{21})^2 + A_{22}A_{23} + A_{23}A_{22} = 0. \quad (13)$$

Now, equations (12) and (13) give us $A_{21}^* = (-A_{21})^3$.

On the other hand, using the condition $\alpha(\lambda_{a_2})\alpha(\lambda_{a_2^{-1}}) = \lambda_e \otimes 1_{\mathbb{Q}}$, we deduce that

$$A_{23}A_{21}^* = A_{22}A_{21}^* = 0. \quad (14)$$

Hence,

$$\begin{aligned} A_{21}^*A_{21}^* &= A_{21}(A_{22}A_{23} + A_{23}A_{22})A_{21}^* \text{ (using (12))} \\ &= (A_{21}A_{22})(A_{23}A_{21}^*) + (A_{21}A_{23})(A_{22}A_{21}^*) \\ &= 0 \text{ (by (14)).} \end{aligned}$$

Thus, the equality $A_{21}^* = (-A_{21})^3$ implies $A_{21}^* = A_{21} = 0$ and also we have $A_{12} = A_{13} = 0$ applying κ . This reduces the fundamental unitary to

$$\begin{pmatrix} A_{11} & 0 & 0 & \cdots & A_{1(2l-2)} & A_{1(2l-1)} \\ 0 & A_{22} & A_{23} & \cdots & A_{2(2l-2)} & A_{2(2l-1)} \\ 0 & A_{23}^* & A_{22}^* & \cdots & A_{2(2l-1)}^* & A_{2(2l-2)}^* \\ \vdots & & \vdots & & & \\ A_{l1} & A_{l2} & A_{l3} & \cdots & A_{l(2l-2)} & A_{l(2l-1)} \\ A_{l1}^* & A_{l2}^* & A_{l3}^* & \cdots & A_{l(2l-1)}^* & A_{l(2l-2)}^* \end{pmatrix}.$$

The rest of the proof will be similar to Theorems 3.6 and 3.4, hence omitted. \square

Remark 3.8 *The above results allow us to compute the cases $\Gamma = \underbrace{(\mathbb{Z}_{n_1} * \mathbb{Z}_{n_1} \cdots * \mathbb{Z}_{n_1})}_{k_1 \text{ copies}}$ $* \underbrace{(\mathbb{Z}_{n_2} * \mathbb{Z}_{n_2} \cdots * \mathbb{Z}_{n_2})}_{k_2 \text{ copies}} * \cdots * \underbrace{(\mathbb{Z}_{n_l} * \mathbb{Z}_{n_l} \cdots * \mathbb{Z}_{n_l})}_{k_l \text{ copies}}$ where $n_1 \neq n_2 \neq \cdots \neq n_l$ except the case $n_1 = 2, n_l = \infty$. From [4] we also came to know about $\mathbb{Q} \underbrace{(\mathbb{Z}_n * \mathbb{Z}_n \cdots * \mathbb{Z}_n)}_{k \text{ copies}}$ except the case $n = 4$. We discuss the $n = 4$ case in [18].*

4 Computations for direct product of cyclic groups

Let us make the convention of calling $\mathbb{Q}(\Gamma)$ commutative or non-commutative if the underlying C^* -algebra is commutative or non-commutative respectively. In this section we will give a necessary and sufficient condition on Γ such that $\mathbb{Q}(\Gamma)$ is commutative.

Theorem 4.1 If $\Gamma = \underbrace{(\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n)}_{k \text{ copies}}$ where $n \neq 2, 4$ then $\mathbb{Q}(\Gamma) \cong C(\text{ISO}(\Gamma))$.

Proof:

For simplicity we present the case $k = 2$. Let $\Gamma = \langle a, b \rangle$ with $o(a) = o(b) = n$. The general case will follow by using similar arguments. Write the fundamental unitary as

$$U = \begin{pmatrix} A & B & C & D \\ B^* & A^* & D^* & C^* \\ E & F & G & H \\ F^* & E^* & H^* & G^* \end{pmatrix} = ((u_{ij})).$$

We need a few lemmas to proceed further.

Lemma 4.2 $AB = BA = CD = DC = EF = FE = GH = HG = 0$.

Proof:

Using the relation $\alpha(\lambda_a)\alpha(\lambda_{a^{-1}}) = \alpha(\lambda_{a^{-1}})\alpha(\lambda_a) = \lambda_e \otimes 1_{\mathbb{Q}}$ and comparing the coefficients of $\lambda_{a^2}, \lambda_{a^{-2}}, \lambda_{b^2}, \lambda_{b^{-2}}$ we have

$$AB^* = BA^* = B^*A = A^*B = CD^* = DC^* = C^*D = D^*C = 0.$$

Applying the antipode on these relations one can deduce

$$AB = BA = EF = FE = 0.$$

By similar arguments using the condition $\alpha(\lambda_b)\alpha(\lambda_{b^{-1}}) = \alpha(\lambda_{b^{-1}})\alpha(\lambda_b) = \lambda_e \otimes 1_{\mathbb{Q}}$ we get $CD = DC = GH = HG = 0$. \square

Lemma 4.3 Product of any two different elements of each row and column of the unitary U is zero. i.e. $u_{ij}u_{ik} = 0 \ \forall \ j \neq k$ and $u_{ji}u_{ki} = 0 \ \forall \ j \neq k$.

Proof:

From the condition $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ one can deduce $AE = EA$, comparing the coefficient of λ_{a^2} on both sides. Hence $C^*A^* = A^*C^*$, which shows that $AC = CA$ (applying κ and then taking adjoint). Further, using $\alpha(\lambda_a)\alpha(\lambda_{a^{-1}}) = \lambda_e \otimes 1_{\mathbb{Q}}$ comparing the coefficient of $\lambda_{ab^{-1}}$ on both sides one can get $AC^* + DB^* = 0$. We have

$$\begin{aligned} AC^*A^* &= -DB^*A^* \\ &= -D(AB)^* \\ &= 0 \text{ (by Lemma 4.2).} \end{aligned}$$

Now,

$$\begin{aligned}
(AC)(AC)^* &= (AC)(C^*A^*) \\
&= (CA)(C^*A^*) \text{ (as } AC = CA) \\
&= 0 \text{ (as } AC^*A^* = 0).
\end{aligned}$$

Hence, $AC = CA = 0$ and $AE = EA = 0$ (taking κ and $*$).

Moreover, comparing the coefficient of λ_{a^2} in $\alpha(\lambda_{ab^{-1}}) = \alpha(\lambda_{b^{-1}a})$ we get $AF^* = F^*A$ and $AD = DA$ applying the antipode and adjoint respectively. We next obtain $AD^* + CB^* = 0$ comparing the coefficient of λ_{ab} on both sides of the equation $\alpha(\lambda_a)\alpha(\lambda_{a^{-1}}) = \lambda_e \otimes 1_{\mathbb{Q}}$, which implies $AD^*A^* = 0$ as $B^*A^* = 0$. Furthermore,

$$\begin{aligned}
(AD)(AD)^* &= (AD)(D^*A^*) \\
&= (DA)(D^*A^*) \text{ (as } AD = DA) \\
&= 0 \text{ (as } AD^*A^* = 0).
\end{aligned}$$

Thus $AD = DA = 0$ and $AF^* = F^*A = 0$ (taking κ and $*$).

Next, we want to show that $BD = DB = BC = CB = 0$. From $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ and comparing the coefficient of $\lambda_{a^{-2}}$ we deduce $BF = FB$, which gives $DB = BD$ (taking κ) and $BD^* + CA^* = 0$, implying $BD^*B^* = 0$. Then $(BD)(BD)^* = D(BD^*B^*) = 0$ as $BD = DB$ and $BD^*B^* = 0$. So we get $BD = DB = 0$. Similarly, we have $BC = CB = 0$.

We also obtain similar relations replacing A, B, C, D by E, F, G, H respectively. \square .

Lemma 4.4 *All the entries of U are normal and partial isometries. i.e. $u_{ij}u_{ij}^* = u_{ij}^*u_{ij}$, $u_{ij}u_{ij}^*u_{ij} = u_{ij} \forall i, j$.*

Proof:

The unitarity of U gives us

$$AA^* + BB^* + CC^* + DD^* = 1,$$

$$A^*A + B^*B + C^*C + D^*D = 1.$$

Using Lemma 4.3 we have $A^*A^2 = A$ and $A(A^*)^2 = A^*$. Now $AA^* = A^*A^2A^* = A^*A$, i.e. A is a normal element. We can prove normality of other elements by similar arguments. Now we are going to show that all of them are partial isometries. First note that $AB^* = 0$ by Lemma 4.2 and Proposition 2.20. We claim $AC^* = AD^* = 0$ too, which will imply $AA^*A = A$, multiplying by A on the left side of equation $A^*A + B^*B + C^*C + D^*D = 1$. Moreover,

$$\begin{aligned}
(AC^*)(AC^*)^* &= AC^*CA^* \\
&= ACC^*A^* \text{ (as } CC^* = C^*C) \\
&= 0 \text{ (as } AC = 0 \text{ by Lemma 4.3)}.
\end{aligned}$$

This shows that $AC^* = 0$. By the same argument we can deduce $AD^* = 0$. Similar arguments will work for the other elements. \square

Lemma 4.5 *A and B commute with each elements of the set $\{G, H, G^*, H^*\}$. Similarly, E and F commute with each elements of the set $\{C, D, C^*, D^*\}$.*

Proof:

From the relation $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ comparing the coefficient of λ_{ab} on both sides we obtain

$$AG + CE = GA + EC. \quad (15)$$

This gives $A^2G + ACE = AGA + AEC$, multiplying by A on the left side of (15), hence $A^2G = AGA$ as $AC = AE = 0$ by Lemma 4.3.

On the other hand, multiplying by A on the right side of (15) we get $AGA = GA^2$. Thus, G commutes with A^2 and

$$A^2G = GA^2 = AGA. \quad (16)$$

Moreover, G commutes with $(A^*)^2$ also by Lemma 4.4. Now,

$$\begin{aligned} AG &= A^*A^2G \text{ (using Lemma 4.4)} \\ &= A^*AGA \text{ (by (16))} \\ &= A^2(A^*)^2GA \text{ (by Lemma 4.4)} \\ &= GA^2(A^*)^2A \text{ (as G commutes with } A^2 \text{ and } (A^*)^2) \\ &= GAA^*A \text{ (using Lemma 4.4)} \\ &= GA. \end{aligned}$$

We obtain the remaining commutation relations in a similar way. \square

By the above lemmas together with Proposition 2.30, the proof of Theorem 4.1 is completed. \square

Remark 4.6 *From the proof it is easily seen that we can take also $n = \infty$. For finite n we get one extra relation $u_{ij}^* = u_{ij}^{n-1}$, where $((u_{ij}))$ denotes the fundamental unitary. A close look at the proof will reveal that the fact a^2, a^{-2}, b^2, b^{-2} are different elements plays a crucial role here. For this reason it does not work for the cases $n = 2, 4$.*

Theorem 4.7 *Let $\Gamma = \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_l$ where $\Gamma_i = \underbrace{(\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i} \cdots \times \mathbb{Z}_{n_i})}_{k_i \text{ copies}}$.*

Also assume that $n_1 \neq n_2 \neq \cdots \neq n_l$ and $n_i \neq 2, 4 \forall i$, then $\mathbb{Q}(\Gamma)$ will be

$$C(\underbrace{ISO(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \cdots \times \mathbb{Z}_{n_1})}_{k_1 \text{ copies}}) \hat{\otimes} C(\underbrace{ISO(\mathbb{Z}_{n_2} \times \mathbb{Z}_{n_2} \cdots \times \mathbb{Z}_{n_2})}_{k_2 \text{ copies}}) \hat{\otimes} \cdots \hat{\otimes} C(\underbrace{ISO(\mathbb{Z}_{n_l} \times \mathbb{Z}_{n_l} \cdots \times \mathbb{Z}_{n_l})}_{k_l \text{ copies}}).$$

i.e. $\mathbb{Q}(\Gamma) \cong \mathbb{Q}(\Gamma_1) \hat{\otimes} \mathbb{Q}(\Gamma_2) \hat{\otimes} \cdots \hat{\otimes} \mathbb{Q}(\Gamma_l)$.

Proof:

As before we give the proof for the case where all $k_i = 1$ to simplify notation.

Now $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l}$ and choose $\{a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_l, a_l^{-1}\}$ as the standard symmetric generating set of Γ , where $o(a_i) = n_i \forall i$.

Fundamental unitary is of the form

$$U = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2l-1)} & A_{1(2l)} \\ A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \cdots & A_{1(2l)}^* & A_{1(2l-1)}^* \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\ A_{22}^* & A_{21}^* & A_{24}^* & A_{23}^* & \cdots & A_{2(2l)}^* & A_{2(2l-1)}^* \\ \vdots & & & \vdots & & & \\ A_{l1} & A_{l2} & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\ A_{l2}^* & A_{l1}^* & A_{l4}^* & A_{l3}^* & \cdots & A_{l(2l)}^* & A_{l(2l-1)}^* \end{pmatrix}.$$

First we want to reduce it to a block diagonal form. By Remark 2.26 this will complete the proof, because in this case $\mathbb{Q}(\Gamma_i)$ is commutative by Theorem 4.1, hence \otimes^{max} coincides with $\hat{\otimes}$.

We remark that

$$A_{1i}A_{1j} = A_{1j}A_{1i} = 0 \forall i \neq j \quad (17)$$

by arguments similar to those in the proof of Lemma 4.3. Using (17) we have $\alpha(\lambda_{a_1^{n_1-1}}) = \lambda_{a_1^{n_1-1}} \otimes A_{11}^{n_1-1} + \lambda_{(a_1^{-1})^{n_1-1}} \otimes A_{12}^{n_1-1} + \cdots + \lambda_{a_l^{n_1-1}} \otimes A_{1(2l-1)}^{n_1-1} + \lambda_{(a_l^{-1})^{n_1-1}} \otimes A_{1(2l)}^{n_1-1}$. Now, from the relation $\alpha(\lambda_{a_1^{n_1-1}}) = \alpha(\lambda_{a_1^{-1}})$ one obtains

$$A_{12}^* = A_{12}^{n_1-1}, A_{11}^* = A_{11}^{n_1-1}, A_{1i}^* = 0 \forall i > 2 \quad (18)$$

by comparing the coefficients of $\lambda_{a_1}, \lambda_{a_1^{-1}}, \lambda_{a_2}, \lambda_{a_2^{-1}}, \dots, \lambda_{a_l}, \lambda_{a_l^{-1}}$. Applying the antipode on (18) we find $A_{i1} = A_{i2} = 0 \forall i > 1$. Now the fundamental unitary reduces to

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\ A_{12}^* & A_{11}^* & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\ 0 & 0 & A_{24}^* & A_{23}^* & \cdots & A_{2(2l)}^* & A_{2(2l-1)}^* \\ \vdots & & & \vdots & & & \\ 0 & 0 & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\ 0 & 0 & A_{l4}^* & A_{l3}^* & \cdots & A_{l(2l)}^* & A_{l(2l-1)}^* \end{pmatrix}.$$

Proceeding in a similar way we get the desired block diagonal form. Note that the last relation $\alpha(\lambda_{a_l^{n_l-1}}) = \alpha(\lambda_{a_l^{-1}})$ has not been used. For this reason we can also include the case $n_l = \infty$. \square

Theorem 4.8 *The conclusion of Theorem 4.7 remains valid if $n_1 = 2$ where $2 < n_2 < n_3 \cdots < n_l \leq \infty$ and $n_i \neq 4 \forall i$.*

Proof:

For simplicity consider $k_i = 1 \forall i$ so that $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \times \cdots \times \mathbb{Z}_{n_l}$ and $o(a_1) = 2$.

We only show how to reduce the fundamental unitary to the form given below, rest of the arguments are similar to those of Theorem 4.7.

$$\begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_{22} & A_{23} & \cdots & 0 & 0 \\ 0 & A_{23}^* & A_{22}^* & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & A_{l(2l-2)} & A_{l(2l-1)} \\ 0 & 0 & 0 & \cdots & A_{l(2l-1)}^* & A_{l(2l-2)}^* \end{pmatrix}$$

Note that product of any two different elements of each row is zero (except possibly the first row) following the lines of arguments of Lemma 4.3, which means

$$A_{k1}A_{ki} = 0 \forall k, i > 1. \quad (19)$$

Furthermore, considering the coefficient of λ_e in the expression of $\alpha(\lambda_{a_k^2})$ we obtain

$$A_{k1}^2 = 0 \forall k > 1. \quad (20)$$

Our aim is to show that $A_{k1} = 0 \forall k > 1$. Now, using the condition $\alpha(\lambda_{a_k})\alpha(\lambda_{a_k^{-1}}) = \lambda_e \otimes 1_{\mathbb{Q}} \forall k > 1$, one can deduce

$$A_{k1}A_{k1}^* + A_{k2}A_{k2}^* + \cdots + A_{k(2l-1)}A_{k(2l-1)}^* = 1. \quad (21)$$

Multiplying by A_{k1} on the left side of (21) we get

$$A_{k1}(A_{k1}A_{k1}^* + A_{k2}A_{k2}^* + \cdots + A_{k(2l-1)}A_{k(2l-1)}^*) = A_{k1} \forall k > 1. \quad (22)$$

Now using (19) and (20) one can find $A_{k1} = A_{k1}^* = 0 \forall k > 1$, hence $A_{12} = A_{13} = A_{14} = A_{15} = \cdots A_{1(2l-2)} = A_{1(2l-1)} = 0$ by applying the antipode.

This gives one-step reduction of the fundamental unitary to the following form

$$U = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_{22} & A_{23} & \cdots & A_{2(2l-2)} & A_{2(2l-1)} \\ 0 & A_{23}^* & A_{22}^* & \cdots & A_{2(2l-1)}^* & A_{2(2l-2)}^* \\ \vdots & & & \ddots & & \\ 0 & A_{l2} & A_{l3} & \cdots & A_{l(2l-2)} & A_{l(2l-1)} \\ 0 & A_{l3}^* & A_{l2}^* & \cdots & A_{l(2l-1)}^* & A_{l(2l-2)}^* \end{pmatrix}.$$

Then we proceed similarly as Theorem 4.7 to achieve the desired reduction .
□

Remark 4.9 Notice that unlike the free case, the above proof even works for the case when $o(a_l) = \infty$.

Theorem 4.10 The conclusion of Theorem 4.7 is valid for $n_1 = 4$ where $4 \neq n_2 \neq \dots \neq n_l$ and $n_i \neq 2 \forall i$, if we replace $\hat{\otimes}$ by \otimes^{max} , i.e. $\mathbb{Q}(\Gamma) \cong \mathbb{Q}(\Gamma_1) \otimes^{max} \mathbb{Q}(\Gamma_2) \otimes^{max} \dots \otimes^{max} \mathbb{Q}(\Gamma_l)$. However, in this case $\mathbb{Q}(\Gamma)$ is non-commutative.

Proof:

As before, assume without loss of generality and for the sake of simplicity of exposition that $k_i = 1 \forall i$ and $o(a_1) = 4$. Again, the product of any two different elements of each row and column is zero (except the first two rows and columns) by the arguments of Lemma 4.3. Hence all entries except possibly the first two rows and columns are normal by arguments similar to those of Lemma 4.4. Now we claim that $A_{k1} = A_{k2} = 0 \forall k > 1$. Considering the coefficients of $\lambda_{a_1}, \lambda_{a_1^{-1}}$ of $\alpha(\lambda_{a_k^3}) \forall k > 1$ we have

$$(A_{k1}^2 + A_{k2}^2)A_{k1} = (A_{k1}^2 + A_{k2}^2)A_{k2} = 0. \quad (23)$$

This gives us $A_{k1}^3 = A_{k2}^3 = 0$, hence $A_{k1} = A_{k2} = 0 \forall k > 1$, as they are normal. Applying the antipode we deduce $A_{1k} = 0 \forall k > 2$.

Rest of the proof is very similar to Theorem 4.7, hence omitted. \square

Combining the above theorems in this section and Propositions 2.18 and 2.21 we get the following necessary and sufficient condition for $\mathbb{Q}(\Gamma)$ to be commutative.

Corollary 4.11 $\mathbb{Q}(\Gamma)$ is commutative if and only if Γ must be of the form $\underbrace{(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \dots \times \mathbb{Z}_{n_1})}_{k_1 \text{ copies}} \times \underbrace{(\mathbb{Z}_{n_2} \times \mathbb{Z}_{n_2} \dots \times \mathbb{Z}_{n_2})}_{k_2 \text{ copies}} \times \dots \times \underbrace{(\mathbb{Z}_{n_l} \times \mathbb{Z}_{n_l} \dots \times \mathbb{Z}_{n_l})}_{k_l \text{ copies}}$ where $n_i \neq 4 \forall i$ and if $n_j = 2$ for some j , then k_j must be 1.

5 Examples of (Γ, S) for which $\mathbb{Q}(\Gamma) \cong \mathcal{D}_\theta(C^*(\Gamma), \Delta_\Gamma)$

It has been observed in Proposition 2.3 of [21] that, if there exists a non trivial automorphism of order 2 which preserves the generating set, then $\mathcal{D}_\theta(C^*(\Gamma), \Delta_\Gamma)$ is always a quantum subgroup of $\mathbb{Q}(\Gamma)$. For many examples studied by the authors of [7], [13], [22] $\mathbb{Q}(\Gamma)$ coincides with the doubled group algebra. In this section we produce more examples of groups where this occurs.

5.1 Dihedral groups with two different generating sets

Dihedral group has two presentations

$$D_{2n} = \langle a, b \mid a^2 = b^n = e, ab = b^{-1}a \rangle, \quad (24)$$

$$D_{2n} = \langle s, t \mid s^2 = t^2 = (st)^n = e \rangle, \quad (25)$$

where e denotes the identity element of the group. In [22] the authors calculated QISO for $D_{2(2n+1)}$ with the presentation (24). Let us calculate it for D_{2n} with presentation (25).

Theorem 5.1 *Let $D_{2n} = \langle s, t \mid s^2 = t^2 = (st)^n = e \rangle$, then its QISO is isomorphic to $\mathcal{D}_\theta(C^*(D_{2n}), \Delta_{D_{2n}})$ with respect to the automorphism θ given by $\theta(s) = t$, $\theta(t) = s$.*

Proof:

The action is defined by

$$\begin{aligned}\alpha(\lambda_s) &= \lambda_s \otimes A + \lambda_t \otimes B, \\ \alpha(\lambda_t) &= \lambda_s \otimes C + \lambda_t \otimes D.\end{aligned}$$

Here $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the corresponding fundamental unitary.

We present the case for odd $n = 2k + 1$, as the proof is almost the same for even n .

We begin with the simple observation that given $s^2 = t^2 = e$, the condition $(st)^n = e$ is equivalent to $(st)^k s = (ts)^k t$. From the relation $\alpha(\lambda_s) = \alpha(\lambda_{s^{-1}})$ we have $A = A^*$, $B = B^*$ and using $\alpha(\lambda_{s^2}) = \lambda_e \otimes 1_{\mathbb{Q}}$ we get $A^2 + B^2 = 1$, $AB = BA = 0$.

Similarly $C = C^*$, $D = D^*$ and $C^2 + D^2 = 1$, $CD = DC = 0$.

Applying the antipode on the above equations we find that

$$\begin{aligned}A^2 + C^2 &= B^2 + D^2 = 1, \\ AC &= CA = BD = DB = 0.\end{aligned}$$

So we obtain $A^2 = D^2$, $B^2 = C^2$ and clearly A^2, B^2, C^2, D^2 are central projections. Now we are going to use $(st)^n = (st)^{2k+1} = e$, from which we deduce

$$(st)^k s = (ts)^k t. \quad (26)$$

We want to obtain analogues of (26) with (s, t) replaced by (A, D) as well as (B, C) . Using relation (26) we have

$$(AD)^k A + (BC)^k B = (DA)^k D + (CB)^k C. \quad (27)$$

Applying κ on (27) we get

$$(AD)^k A + (CB)^k C = (DA)^k D + (BC)^k B. \quad (28)$$

Equations (27) and (28) together imply $(AD)^k A = (DA)^k D$, $(CB)^k C = (BC)^k B$. Now it follows from Lemma 2.27 that $\mathbb{Q}(D_{2n})$ coincides with $\mathcal{D}_\theta(C^*(D_{2n}), \Delta_{D_{2n}})$ corresponding to the order 2 automorphism θ given by $\theta(s) = t$, $\theta(t) = s$. \square

Remark 5.2 *We get the same result for D_{2n} with presentation (24) except $n = 4$ case. The proof is very similar to the case $\mathbb{Z}_2 \times \mathbb{Z}_n$, hence omitted. This extends the result of [22].*

5.2 Baumslag-Solitar group

The group has the presentation $\Gamma = \langle a, b \mid b^{-1}ab = a^2 \rangle$.

We can easily get the following relations among the generators,

$$\begin{aligned} b^{-1}ab &= a^2, \quad b^{-1}a^{-1}b = a^{-2}, \\ ab &= ba^2, \quad b^{-1}a = a^2b^{-1}, \quad a^{-1}b^{-1}a = ab^{-1}, \\ b^{-1}a^{-1} &= a^{-2}b^{-1}, \quad a^{-1}b = ba^{-2}, \quad ab = ba^2, \quad aba^{-1} = ba. \end{aligned}$$

Write the fundamental unitary as

$$\begin{pmatrix} A & B & C & D \\ B^* & A^* & D^* & C^* \\ E & F & G & H \\ F^* & E^* & H^* & G^* \end{pmatrix}.$$

Our aim is to show $C = D = E = F = 0$.

Using the relation $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba^2})$ and comparing the coefficients of λ_{b^2} and $\lambda_{b^{-2}}$, we obtain $CG = DH = 0$ and by applying the antipode we also get $EG = HF = 0$.

Further, using the condition $\alpha(\lambda_{a^{-1}b^{-1}a}) = \alpha(\lambda_{ab^{-1}})$ one can find $CH^* = DG^* = 0$, hence $EH = GF = 0$. Also note that $EF = 0$ arguing along the lines of Lemma 3.2. So $E(GG^* + HH^* + FF^* + EE^*) = E$, implying $E^2E^* = E$ as $(GG^* + HH^* + FF^* + EE^*) = 1$, $EG = EF = EH = 0$. Similarly, we obtain $F^*F^2 = F$.

Next, we compare the coefficients of λ_{b^2} and $\lambda_{b^{-2}}$ on both sides of $\alpha(\lambda_{b^{-1}ab}) = \alpha(\lambda_{a^2})$ to have $C^2 = D^2 = 0$, which implies $E^2 = F^2 = 0$ using antipode and taking adjoint of the elements.

Now using the above relations, $C = D = E = F = 0$ as $E^2E^* = E$, $F^*F^2 = F$.

The fundamental unitary reduces to the form

$$\begin{pmatrix} A & B & 0 & 0 \\ B^* & A^* & 0 & 0 \\ 0 & 0 & G & H \\ 0 & 0 & H^* & G^* \end{pmatrix}.$$

Moreover, we claim that $H = 0$. Using $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba^2})$ we have

$AH = BH = 0$, equating the coefficients of $\lambda_{ab}, \lambda_{a^{-1}b^{-1}}$ on both sides. Thus $(A^*A + B^*B)H = H$, that is, $H = 0$.

This gives the following reduction:

$$\begin{pmatrix} A & B & 0 & 0 \\ B^* & A^* & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G^* \end{pmatrix}.$$

Moreover, using the relations among the generators we deduce that $G^*AG =$

A^2 , $G^*BG = B^2$, $AG = GA^2$, $BG = GB^2$ and also that AA^* , BB^* are central projections of the algebra.

It now follows from Lemma 2.27 that $\mathbb{Q}(\Gamma)$ is isomorphic to $\mathcal{D}_\theta(C^*(\Gamma), \Delta_\Gamma)$ with respect to the automorphism θ given by $a \mapsto a^{-1}$, $b \mapsto b$. \square

5.3 Some groups of the form $\langle a, b \mid o(a) = 2, o(b) = 3 \rangle$

First we conclude a lemma which will be useful for the proof of Theorem 5.4.

Lemma 5.3 *If $\Gamma = \langle a, b \mid o(a) = 2, o(b) = 3 \rangle$, then its fundamental unitary must be of the form*

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E & F \\ 0 & F^* & E^* \end{pmatrix}.$$

Proof:

Using the relation $\alpha(\lambda_{b^2}) = \alpha(\lambda_{b^{-1}})$, and comparing the coefficients of λ_a and $\lambda_{a^{-1}}$ from both sides we will get the reduced block diagonal form. \square

Now we use the above lemma to compute quantum isometry groups of some concrete examples.

Theorem 5.4 *Let Γ be as in the statement of Lemma 5.3 and consider the automorphism θ defined by $\theta(a) = a$, $\theta(b) = b^{-1}$. Then $\mathbb{Q}(\Gamma) \cong \mathcal{D}_\theta(C^*(\Gamma), \Delta_\Gamma)$, for the following three examples:*

1. $(ab)^3 = 1$, (Tetrahedral)
2. $(ab)^4 = 1$, (Octahedral)
3. $(ab)^5 = 1$. (Icosahedral)

Proof:

In each of these cases, we can apply Lemma 5.3 to get A, E, F such that the action α is given by

$$\begin{aligned} \alpha(\lambda_a) &= \lambda_a \otimes A, \\ \alpha(\lambda_b) &= \lambda_b \otimes E + \lambda_{b^{-1}} \otimes F, \\ \alpha(\lambda_{b^{-1}}) &= \lambda_b \otimes F^* + \lambda_{b^{-1}} \otimes E^*. \end{aligned}$$

Also $A^2 = 1$, $A = A^*$, applying $\alpha(\lambda_a) = \alpha(\lambda_{a^{-1}})$ and $\alpha(\lambda_{a^2}) = \lambda_e \otimes 1_{\mathbb{Q}}$. Similarly $E^2 = E^*$, $F^2 = F^*$ using the condition $\alpha(\lambda_{b^2}) = \alpha(\lambda_{b^{-1}})$. We also get $EF = FE = 0$ arguing along the lines of Lemma 4.2.

Now consider the relation $(ab)^n = 1$ ($n = 3, 4, 5$ respectively), which gives $ab = (b^{-1}a)^{n-1}$ and $ba = (ab^{-1})^{n-1}$. Using these relations we can deduce

$$AE = \underbrace{(E^*A)(E^*A) \cdots (E^*A)}_{(n-1) \text{ times}}, \quad (29)$$

$$EA = \underbrace{(AE^*)(AE^*) \cdots (AE^*)}_{(n-1) \text{ times}}. \quad (30)$$

Moreover,

$$\begin{aligned} A(E E^*) &= \underbrace{(E^*A)(E^*A) \cdots (E^*A)}_{(n-1) \text{ times}} E^* \text{ (by (29))} \\ &= E^* \underbrace{(AE^*)(AE^*) \cdots (AE^*)}_{(n-1) \text{ times}} \\ &= E^*(EA) \text{ (using (30))} \\ &= (EE^*)A \text{ (as } E^* = E^2). \end{aligned}$$

Hence, EE^* is a central projection. By similar arguments it can be proved that FF^* is a central projection. By Lemma 2.27 we get the isomorphism between $\mathbb{Q}(\Gamma)$ and $\mathcal{D}_\theta(C^*(\Gamma), \Delta_\Gamma)$ with respect to the automorphism θ given by $b \mapsto b^{-1}$, $a \mapsto a$. \square

6 Coxeter groups as examples of Γ such that $\mathbb{Q}(\Gamma) \cong (C^*(\Gamma), \Delta_\Gamma)$

In this section we will compute QISO for certain classes of Coxeter groups. The Coxeter group with parameters (l, m, n) and $l \leq m \leq n$ has the following presentation

$$\Gamma = \langle a, b, c \mid o(a) = o(b) = o(c) = 2, (ac)^l = (ab)^m = (bc)^n = e \rangle.$$

Here we take one special class, namely $l = 2, m = 3$ and n is any positive integer co-prime to 6.

Theorem 6.1 *Let Γ be the Coxeter group with parameters $(2, 3, n)$ as above. Then $\mathbb{Q}(\Gamma) \cong (C^*(\Gamma), \Delta_\Gamma)$.*

Proof:

The fundamental unitary is of the form

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}.$$

We divide the proof into a number of lemmas.

Lemma 6.2 $B = D = H = F = 0$.

Proof:

First note that $\alpha(\lambda_{ac}) = \alpha(\lambda_{ca})$ as $(ac)^2 = a^2 = c^2 = e$.

$$\alpha(\lambda_{ac}) = \lambda_e \otimes (AG + BH + CK) + \lambda_{ac} \otimes (AK + CG) + \lambda_{ab} \otimes AH + \lambda_{bc} \otimes BK + \lambda_{ba} \otimes BG + \lambda_{cb} \otimes CH,$$

$$\alpha(\lambda_{ca}) = \lambda_e \otimes (GA + HB + KC) + \lambda_{ac} \otimes (KA + GC) + \lambda_{ab} \otimes GB + \lambda_{bc} \otimes HC + \lambda_{ba} \otimes HA + \lambda_{cb} \otimes KB.$$

From the above equations we have

$$AH = GB, \quad BK = HC, \quad HA = BG, \quad CH = KB.$$

Now as the action is length preserving, we also get

$$AG + BH + CK = 0. \quad (31)$$

Using the condition $\alpha(\lambda_{b^2}) = \alpha(\lambda_e)$ one obtains $DF + FD = 0$ by comparing the coefficient of λ_{ac} on both sides, and hence, $HB + BH = 0$ (applying κ).

This gives us

$$HBH + BH^2 = 0. \quad (32)$$

Again using $\alpha(\lambda_{c^2}) = \alpha(\lambda_e)$ we have $G^2 + H^2 + K^2 = 1$, hence

$$BG^2 + BH^2 + BK^2 = B. \quad (33)$$

Now (31) implies $HAG + HBH + HCK = 0$, which gives $(BG)G + HBH + (BK)K = 0$, as $HA = BG$, $BK = HC$. Thus we get $BG^2 - BH^2 + BK^2 = 0$ using (32). Comparing it with (33) we deduce $B = 2BH^2$.

We will show now $BH^2 = 0$. Multiplying H on the right side of (31) we find $AGH + BH^2 + CKH = 0$, where $GH = KH = 0$, using $\alpha(\lambda_{c^2}) = \alpha(\lambda_e)$ and comparing the coefficients of λ_{ab} , λ_{bc} respectively.

So $BH^2 = 0$, hence $B = 0$, and $D = 0$ too applying the antipode.

This also gives $HA = BG = 0$, $HC = BK = 0$. Moreover, $(A^2 + C^2) = 1$ by using $\alpha(\lambda_{a^2}) = \alpha(\lambda_e)$. We have

$$\begin{aligned} H &= H(A^2 + C^2) \quad (as \ (A^2 + C^2) = 1) \\ &= (HA)A + (HC)C \\ &= 0 \quad (as \ HA = HC = 0). \end{aligned}$$

Similarly, one can get $F = 0$. \square

Lemma 6.3 $G = C = 0$.

Proof:

First we need to derive some more relations among the generators from the defining ones.

From $(ab)^3 = e$ we have $aba = bab$ as $a^2 = b^2 = e$. Our aim is to show that $ECE = 0$. We claim that the term $bc b$ is not equal to any of the terms aba, abc, cba, cbc . Clearly, $bc b \neq aba$ as we have $aba = bab$. Now $cbc \neq bcb$ as $(bc)^3 \neq e$. If $(bc)^3 = e$ then using the hypothesis we can obtain $bc = e$ as n is co-prime to 3 too, which implies $b = c$. Furthermore, if $bc b = abc$ which implies $(bc)^2 = ab$, then we get $(bc)^6 = e$. Hence, one can deduce $bc = e$. Similarly, we can argue that $bc b$ can't be equal to cba . Now using $\alpha(\lambda_{bab}) = \alpha(\lambda_{aba})$ we obtain

$$\alpha(\lambda_{bab}) = \lambda_{bab} \otimes EAE + \lambda_{bcb} \otimes ECE, \quad (34)$$

$$\alpha(\lambda_{aba}) = \lambda_{aba} \otimes AEA + \lambda_{abc} \otimes AEC + \lambda_{cba} \otimes CEA + \lambda_{cbc} \otimes CEC. \quad (35)$$

Comparing the both sides of (34) and (35) we find $ECE = 0$, also $E^2 = 1$ from the condition $\alpha(\lambda_{b^2}) = \alpha(\lambda_e)$. Moreover,

$$\begin{aligned} C &= E^2 C E^2 \text{ (as } E^2 = 1) \\ &= E(ECE)E \\ &= 0 \text{ (as } ECE = 0). \end{aligned}$$

Applying the antipode we find $G = 0$. \square

Proof of Theorem 6.1:

By the above lemmas, we have reduced the fundamental unitary to the following form:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}.$$

So by Corollary 2.19, there is a CQG isomorphism from $(C^*(\Gamma), \Delta_\Gamma)$ to $\mathbb{Q}(\Gamma)$ sending $\lambda_a, \lambda_b, \lambda_c$ to A, E, K respectively. \square

7 An excursion to QISO of compact matrix quantum groups

In this very brief last section, we want to extend the formulation of quantum isometry group to the realm of quantum groups. Let us consider a compact matrix quantum group (\mathcal{Q}, Δ) which has a finite (say n) dimensional unitary fundamental representation π with $((\pi_{ij}))$ being the corresponding unitary in $M_n(\mathcal{Q})$. Indeed, by definition, every irreducible representation of (\mathcal{Q}, Δ) is a sub-representation of tensor copies of π and $\bar{\pi}$, so as in Subsection 2.6, we may consider a central length function l which takes an irreducible say α to the

smallest non-negative integer k such that $\alpha \subset \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k$ where each α_i is either π or $\bar{\pi}$. As shown in [5], this gives rise to a spectral triple, generalizing the construction of D_Γ for a finitely generated discrete group in Subsection 2.1. Moreover, this spectral triple satisfies the condition of Theorem 2.11, hence the quantum isometry group exists. Let us denote the quantum isometry group by $\mathbb{Q}(\hat{\mathcal{Q}}, \{u_{ij}\})$, where u_{ij} consist of both π_{ij} 's as well as π_{ij}^* 's, or simply $\mathbb{Q}(\hat{\mathcal{Q}})$ if the matrix elements u_{ij} are understood. By Gram-Schmidt, we convert $\{u_{ij}\}$ to an orthogonal set, say $\{u'_{ij}\}$ with respect to the Haar state of (\mathcal{Q}, Δ) . Indeed, as in the group case, the action of $\mathbb{Q}(\hat{\mathcal{Q}})$, say β , is determined by q_{kl}^{ij} such that

$$\beta(u'_{ij}) = \sum_{kl} u'_{kl} \otimes q_{kl}^{ij}.$$

In other words, the quantum isometry group is generated by q_{kl}^{ij} subject to the relations that make it a unitary and also make the above β a $*$ -homomorphism from \mathcal{Q} to $\mathcal{Q} \hat{\otimes} \mathbb{Q}(\hat{\mathcal{Q}})$.

Remark 7.1 *Note that $\mathbb{Q}(\hat{\mathcal{Q}}, \{u_{ij}\})$ is the universal C^* -algebra generated by q_{kl}^{ij} such that $Q = ((q_{kl}^{ij}))$ is unitary as well as $Q^t E \bar{Q} E^{-1} = E \bar{Q} E^{-1} Q^t = I$, where $E = ((E_{kl}^{ij})) = (((u'_{ij})^*, (u'_{kl})^*))$ and β given above is a C^* -homomorphism on \mathcal{Q} by the similar argument of Proposition 2.16.*

It is interesting to note that the formulation for quantum isometry group for a compact matrix quantum group allows us to consider even the group algebras with a set of generators which are not necessarily of the form δ_g for elements g of the group, i.e. not necessarily group-like elements of the group C^* -algebra. This flexibility of choice can have quite interesting implications for the resulting quantum isometry groups, as illustrated by the example below. We consider the group $\Gamma = \mathbb{Z} \times \mathbb{Z}_2$. It has a natural set of generators consisting of group-like elements as in Theorem 4.8, where the resulting QISO turned out to be $\mathbb{Q}(\mathbb{Z}) \hat{\otimes} \mathbb{Q}(\mathbb{Z}_2)$. It can be identified as the doubling of the group algebra too. However, we can also view it as a matrix quantum group with a fundamental unitary whose entries are not group-like elements. More precisely, note that $C_r^*(\mathbb{Z} \times \mathbb{Z}_2)$ is isomorphic with $C(\mathbb{T}) \oplus C(\mathbb{T})$ as a C^* -algebra, and it can be described as the C^* -algebra $C^*\{\gamma, \gamma'\}$ where γ, γ' denotes the canonical generators of the two copies of $C(\mathbb{T})$. The C^* -algebra is the universal one with two generators satisfying the following relations:

$$\gamma \cdot \gamma^* = \gamma^* \cdot \gamma, \quad \gamma' \gamma'^* = \gamma'^* \gamma', \quad (36)$$

$$\gamma \cdot \gamma' = \gamma' \cdot \gamma = 0, \quad (37)$$

$$\gamma \cdot \gamma^* + \gamma' \cdot \gamma'^* = 1. \quad (38)$$

From the group structure of Γ , it is easy to see that $\mathcal{F} := \{\gamma, \gamma^*, \gamma', (\gamma')^*\}$ gives the matrix coefficients of a 2-dimensional fundamental unitary representation, not consisting of group-like elements. We have the following description of the quantum isometry group for the generating set \mathcal{F} , which is again a doubling, but not of the group algebra itself.

Theorem 7.2 $\mathbb{Q}(C_r^*(\widehat{\mathbb{Z} \times \mathbb{Z}_2}), \mathcal{F})$ is isomorphic with a doubling of the quantum group $\mathbb{Q}(\mathbb{Z}) \star \mathbb{Q}(\mathbb{Z})$ with respect to the order 2 automorphism θ defined by

$$\gamma_1 \mapsto \gamma'_1, \gamma'_1 \mapsto \gamma_1, \gamma_2 \mapsto \gamma'_2, \gamma'_2 \mapsto \gamma_2,$$

where $\gamma_1, \gamma'_1, \gamma_2, \gamma'_2$ generate the underlying C^* -algebra of the CQG $\mathbb{Q}(\mathbb{Z}) \star \mathbb{Q}(\mathbb{Z})$.

Proof: Corresponding to the action of $\mathbb{Q}(C_r^*(\widehat{\mathbb{Z} \times \mathbb{Z}_2}), \mathcal{F})$, the fundamental unitary is

$$\begin{pmatrix} a_1^{11} & a_2^{11} & a_1^{12} & a_2^{12} \\ (a_2^{11})^* & (a_1^{11})^* & (a_2^{12})^* & (a_1^{12})^* \\ a_1^{21} & a_2^{21} & a_1^{22} & a_2^{22} \\ (a_2^{21})^* & (a_1^{21})^* & (a_2^{22})^* & (a_1^{22})^* \end{pmatrix}.$$

We break the proof into a number of steps.

Step 1 : Using the condition $\alpha(\gamma.\gamma^*) = \alpha(\gamma^*.\gamma)$, comparing the coefficients of $\gamma^2, (\gamma^*)^2$ and $\gamma.\gamma^*$ on both sides we find that

$$a_1^{11}(a_2^{11})^* = (a_2^{11})^*a_1^{11}, \quad (39)$$

$$a_2^{11}(a_1^{11})^* = (a_1^{11})^*a_2^{11}, \quad (40)$$

$$a_1^{11}(a_1^{11})^* + a_2^{11}(a_2^{11})^* = (a_2^{11})^*a_2^{11} + (a_1^{11})^*a_1^{11}. \quad (41)$$

Applying the antipode on (41) we obtain

$$a_1^{11}(a_1^{11})^* + (a_2^{11})^*a_2^{11} = a_1^{11}(a_2^{11})^* + (a_1^{11})^*a_1^{11}. \quad (42)$$

Thus one can conclude that both the elements a_1^{11} and a_2^{11} are normal using (41) and (42). Hence the C^* -algebra $C^*\{a_1^{11}, a_2^{11}\}$ is commutative by (39), (40) and Proposition 2.20.

Applying the same argument replacing γ by γ' one can deduce that $C^*\{a_1^{22}, a_2^{22}\}$ is commutative as well.

Moreover, using $\alpha(\gamma.\gamma^*) = \alpha(\gamma^*.\gamma)$, comparing the coefficients of $(\gamma')^2, (\gamma'^*)^2$ and $\gamma'\gamma'^*$ on both sides, we have

$$a_1^{12}(a_2^{12})^* = (a_2^{12})^*a_1^{12}, \quad (43)$$

$$a_2^{12}(a_1^{12})^* = (a_1^{12})^*a_2^{12}, \quad (44)$$

$$a_1^{12}(a_1^{12})^* + a_2^{12}(a_2^{12})^* = (a_2^{12})^*a_2^{12} + (a_1^{12})^*a_1^{12}. \quad (45)$$

Applying κ on (45) we get

$$a_1^{21}(a_1^{21})^* + (a_2^{21})^*a_2^{21} = a_2^{21}(a_2^{21})^* + (a_1^{21})^*a_1^{21}. \quad (46)$$

On the other hand, one find

$$a_1^{21}(a_1^{21})^* + a_2^{21}(a_2^{21})^* = (a_2^{21})^*a_2^{21} + (a_1^{21})^*a_1^{21}, \quad (47)$$

by comparing the coefficient of $\gamma.\gamma^*$ on both sides of $\alpha(\gamma')\alpha(\gamma'^*) = \alpha(\gamma'^*)\alpha(\gamma')$. Now, by (46) and (47), both the elements a_1^{21}, a_2^{21} are normal. Using the antipode a_1^{12}, a_2^{12} are seen to be normal too. Thus, the C^* -algebra generated by $\{a_1^{12}, a_2^{12}\}$ is commutative. Hence, $C^*\{a_1^{21}, a_2^{21}\}$ is commutative as well, by applying the antipode.

Step 2 : From the given fact $\alpha(\gamma.\gamma') = 0$, comparing the coefficients of γ^2 and $(\gamma^*)^2$, we obtain

$$a_1^{11}a_1^{21} = a_2^{11}a_2^{21} = 0. \quad (48)$$

Applying the antipode on (48) and using the Proposition 2.20 we have

$$a_1^{11}a_1^{12} = a_2^{11}a_2^{12} = 0. \quad (49)$$

Now, repeating the similar arguments using the relation $\alpha(\gamma.(\gamma')^*) = 0$ one can easily check that $a_1^{11}a_2^{12} = a_2^{11}a_1^{12} = 0$. It can also be shown that $a_1^{22}a_1^{21} = a_1^{22}a_2^{21} = a_2^{22}a_2^{21} = 0$ by following the same line of arguments.

Step 3 : Comparing the coefficient of γ^2 from the relation $\alpha(\gamma.\gamma^* + \gamma'.\gamma'^*) = 1 \otimes 1$ we have

$$a_1^{11}(a_2^{11})^* + a_1^{21}(a_2^{21})^* = 0. \quad (50)$$

Multiplying by a_2^{11} on the right side of the equation (50) one can get $a_1^{11}(a_2^{11})^*a_2^{11} = 0$ as $(a_2^{21})^*a_2^{11} = 0$. Hence $a_1^{11}(a_2^{11})^* = 0$, which shows that $a_1^{11}a_2^{11} = 0$ by the Proposition 2.20 and also $a_1^{21}a_2^{21} = 0$. Applying the same argument and comparing $(\gamma')^2$ from the relation $\alpha(\gamma.\gamma^* + \gamma'.\gamma'^*) = 1 \otimes 1$ one can show that $a_1^{12}a_2^{12} = a_1^{22}a_2^{22} = 0$.

Note that the underlying C^* -algebra of $\mathbb{Q}(\mathbb{Z}) \star \mathbb{Q}(\mathbb{Z})$ is isomorphic to $(C(\mathbb{T}) \oplus C(\mathbb{T})) \star (C(\mathbb{T}) \oplus C(\mathbb{T}))$ which is the same as $C^*\{\gamma_1, \gamma_2\} \star C^*\{\gamma'_1, \gamma'_2\}$, where $\{\gamma_1, \gamma_2\}$ and $\{\gamma'_1, \gamma'_2\}$ satisfy the relations (36), (37) and (38). Finally, using the above steps we define the C^* -isomorphism from $(\mathbb{Q}(\mathbb{Z}) \star \mathbb{Q}(\mathbb{Z})) \oplus (\mathbb{Q}(\mathbb{Z}) \star \mathbb{Q}(\mathbb{Z}))$ to $\mathbb{Q}(\widehat{C_r^*(\mathbb{Z} \times \mathbb{Z}_2)}, \mathcal{F})$ by

$$(\gamma_i, 0) \mapsto a_i^{11}, (\gamma'_i, 0) \mapsto a_i^{22}, (0, \gamma_i) \mapsto a_i^{21}, (0, \gamma'_i) \mapsto a_i^{12},$$

for $i = 1, 2$. Indeed this is the doubling of $\mathbb{Q}(\mathbb{Z}) \star \mathbb{Q}(\mathbb{Z})$ with respect to the order 2 automorphism θ defined as in the statement of Theorem 7.2. \square

Remark 7.3 As $\mathbb{Q}(\mathbb{Z}) \star \mathbb{Q}(\mathbb{Z})$ is noncommutative, it is clear that the quantum isometry group of $C_r^*(\mathbb{Z} \times \mathbb{Z}_2)$ with the new generating set \mathcal{F} differs from the previous one, calculated in Theorem 4.8 with group-like generating elements. Moreover, $\mathbb{Q}(\widehat{C_r^*(\mathbb{Z} \times \mathbb{Z}_2)}, \mathcal{F})$ can also be identified with the quantum group K_2^+ (for more details see Section 5 of [2]). We also give yet another description of K_n^+ in [18].

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